MA 408 - Tutorials

Tutorial-1 :

1. X, Y s.t $X \neq \phi, Y \neq \phi$ $X \subseteq Y \subseteq \mathcal{P}(\Omega)$ $\Omega \neq \phi$

① $\mathcal{A}(X) \subseteq \mathcal{A}(Y)$

as

$$\mathcal{A}(X) = \bigcap_{\substack{A \text{ s.t algebra} \\ X \subseteq A}} A \qquad (X \subseteq M \text{ then } \mathcal{A}(X) \subseteq M)$$

$$\mathcal{A}(Y) = \bigcap_{\substack{B \text{ s.t algebra} \\ Y \subseteq B}} B \Rightarrow \mathcal{A}(X) \subseteq \mathcal{A}(Y)$$

as $X \subseteq Y \subseteq B$ so, $\cap A \subseteq \cap B$

similarly $\mathcal{F}_1(X) \subseteq \mathcal{F}_1(Y)$

③ $X \subseteq \mathcal{A}(X)$ $\mathcal{F}_1(X) \subseteq \mathcal{F}_1(\mathcal{A}(X))$ ——— ①

every σ-algebra also algebra

$X \subseteq \mathcal{F}_1(X)$

$\Rightarrow \mathcal{A}(X) \subseteq \mathcal{F}_1(X)$

$\Rightarrow \mathcal{F}_1(\mathcal{A}(X)) \subseteq \mathcal{F}_1(\mathcal{F}_1(X)) = \mathcal{F}_1(X)$ ——— ②

①, ② $\Rightarrow \mathcal{F}_1(X) = \mathcal{F}_1(\mathcal{A}(X))$

3. $\Omega = \mathbb{N}$ $\mathcal{A} = \mathcal{P}(\mathbb{N})$

$\mu : \mathcal{A} \longrightarrow [0, \infty]$

$$\mu(A) = \begin{cases} 0 ; & A \text{ is finite} \\ \infty ; & \text{otherwise} \end{cases}$$

$A = \bigcup_{i=1}^{n} A_i$ s.t $A_i \cap A_j = \phi$ if $i \neq j$

if $\mu(A) = \infty$ then $\exists i$ s.t A_i is infinite, as A_i's are finitely many, if all A_is have finite elements then A was finite element

$\Rightarrow \mu(A_i) = \infty$

so, $\sum_{i=1}^{\infty} \mu(A_i) = \infty = \mu(A)$

if A_i's are finite $\Rightarrow A$ is finite

$\Rightarrow \mu(A) = 0$

$\Rightarrow \mu(A) = \sum_{i=1}^{n} \mu(A_i)$

$\mathbb{N} = \bigcup_{n \in \mathbb{N}} \{n\}$ $\mu(\mathbb{N}) \neq \sum_{n \in \mathbb{N}} \mu(n)$ so μ is not σ-additive

$E_1 \supseteq E_2 \supseteq E_3 \ldots$

s.t $\cap E_i = E$ if $\exists n_0$ s.t $\mu(E_{n_0}) < \infty$

then

$\mu(E_{n_0+k}) = 0 \quad \forall k \in \mathbb{N}$

$\Rightarrow \lim_{k \to \infty} \mu(E_{n_0+k}) = \lim_{n \to \infty} \mu(E_n) = 0 = \mu(E)$

as E_{n_0} finite, E is finite

2. $\Omega = \mathbb{R}$ $C = \{(a,b] \mid -\infty \le a < b < \infty\}$

$\lambda : C \to [0,\infty]$

$$\lambda((a,b]) = \begin{cases} \infty & ; a = -\infty \\ b-a & ; a,b \in \mathbb{R} \\ \infty & ; b = \infty \end{cases}$$

$\lambda(\phi) = 0$ ($\because a=0, b=0$ case)

$I = \bigcup_{i=1}^{\infty} I_i$ $I_i \in C$, $I_i \cap I_j = \phi$ for $i \neq j$

let $I \in \mathcal{I}$ then $\exists$ a finite interval s.t $I \subset \bigcup_i I_i$

$\Rightarrow \mu(I) \le \sum \lambda(I_i)$

if we show $I = \bigcup_i I_i$ I is finite then $\lambda(I) = \sum \lambda(I_i)$

as $\mu(I) < \sum_i \mu(I_i)$

$a = a_1 < b_1 = a_2 < b_2 = a_3 < \cdots < b_{n-1} = a_n < b$

$\sum_{i=1}^{n} b_i - a_i = \sum_{i=1}^{n} \lambda(I_i) < b - a = \lambda(I)$

$\Rightarrow \sum_{i=1}^{n} \lambda(I_i) \le \lambda(I) \Rightarrow \lambda(I) = \sum_{i=1}^{\infty} \lambda(I_i)$

for infinite case $\lambda(I) \le \sum \lambda(I)$ trivial and

$\lambda(I) = \infty$

$\Rightarrow \sum \lambda(I) = \infty$

1. $X, Y \neq \phi$ $X \subseteq Y \subseteq \sigma(\Omega)$ $\Omega \neq \phi$

To prove: $\mathcal{A}(X) \subseteq \mathcal{A}(Y)$ and $\mathcal{F}_1(X) \subseteq \mathcal{F}_1(Y)$, also $\mathcal{F}_1(\mathcal{A}(X)) = \mathcal{F}_1(X)$

proof:

$$\mathcal{A}(X) = \bigcap_{X \subseteq A} A \subseteq B \quad \forall\ X \subseteq Y \subseteq B$$

A is an algebra

B is an algebra containing Y

$\Rightarrow \mathcal{A}(X) \subseteq \bigcap_{Y \subseteq B} B$

$\Rightarrow \mathcal{A}(X) \subseteq \mathcal{A}(Y)$

Similarly,

$$\mathcal{F}_1(X) = \bigcap_{X \subseteq A} A \subseteq B \quad \forall\ X \subseteq Y \subseteq B$$

A is σ-algebra

B is σ-algebra

$\Rightarrow \mathcal{F}_1(X) \subseteq \bigcap_{Y \subseteq B} B = \mathcal{F}_1(Y)$

now as $X \subseteq \mathcal{A}(X)$ ($\because$ condition)

$\Rightarrow \mathcal{F}_1(X) \subseteq \mathcal{F}_1(\mathcal{A}(X))$ ——①

($\because$ seen above)

and $X \subseteq \mathcal{F}_1(X)$ ($\because$ condition)

$\Rightarrow \mathcal{A}(X) \subseteq \mathcal{A}(\mathcal{F}_1(X)) = \mathcal{F}_1(X)$

↑ smallest algebra containing σ-field is itself

$\Rightarrow \mathcal{F}_1(\mathcal{A}(X)) \subseteq \mathcal{F}_1(\mathcal{F}_1(X)) = \mathcal{F}_1(X)$ —— ②

↑ smallest σ-algebra containing itself is $\mathcal{F}_1(X)$

from ①, ②: $\mathcal{F}_1(X) = \mathcal{F}_1(\mathcal{A}(X))$

3. $\Omega = \mathbb{N}$ $\mathcal{A} = \mathcal{P}(\mathbb{N})$

$\mu: \mathcal{A} \longrightarrow [0,\infty]$

$$\mu(A) = \begin{cases} 0 & ; A \text{ is finite} \\ \infty & ; \text{otherwise} \end{cases}$$

$A = \bigcup_{i=1}^{n} A_i \quad A_i \cap A_j = \phi \quad i \neq j$

$\mu(A) = \infty$ then $\exists i$ s.t A_i is infinite, as finite many A_i's but sum ∞

$\Rightarrow \mu(A_i) = \infty$

$\Rightarrow \sum_{i=1}^{n} \mu(A_i) = \mu(A)$

if A_i's finite $\Rightarrow A$ is finite

$\Rightarrow \mu(A) = 0$ n

$\Rightarrow \mu(A) = \sum_{i=1} \mu(A_i)$

μ is not σ-additive as

$\mathbb{N} = \bigcup_{n \in \mathbb{N}} \{n\} \quad \mu(\mathbb{N}) = \infty$ but $\mu(\{n\}) = 0$

$\Rightarrow \sum_{n \in \mathbb{N}} \mu(\{n\}) = 0 \neq \mu(\mathbb{N})$

μ is not σ-additive

now, $E \in A$ s.t $\{E_i\}_{i=1}^{\infty} \subseteq A$, $E_{n+1} \subseteq E_n \ \forall n$ and $\exists n_0$ s.t $\mu(E_{n_0}) < \infty$

$\Rightarrow \mu(E_{n_0}) = 0$

then $\mu(E_{n_0+k}) = 0 \ \forall k \in \mathbb{N}$

$\Rightarrow \lim_{k \to \infty} \mu(E_{n_0+k}) = \lim_{n \to \infty} \mu(E_n) = 0 = \mu(E)$

↑ as E_{n_0} finite $\Rightarrow E$ is finite

2. $\Omega = \mathbb{R}$

$C = \{(a,b] \mid -\infty \leq a \leq b \leq \infty\}$ on $\mathbb{R}$

$\lambda: C \longrightarrow [0,\infty] \qquad \lambda((a,b]) = b-a$ ($a = -\infty$ or $b = \infty \Rightarrow b-a = \infty$)

now trivial $\lambda(\phi) = \lambda((0,0]) = 0$

then let $\{E_i\}_{i=1}^{\infty} \subseteq C$ be disjoint s.t

$$E = \bigcup_{i=1}^{\infty} E_i \in C$$

then E is of form $E = (a,b]$

and

$$E_i \in C \Rightarrow E_i = (a_i, b_i]$$

now

$$E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (a_i, b_i]$$

$$(a,b] = \bigcup_{i=1}^{\infty} (a_i, b_i]$$

then $\varepsilon > 0$ (if $(a,b] \neq \phi$ as otherwise trivial)

s.t $\varepsilon \leq \frac{b-a}{2}$

$$[a+\varepsilon, b] \subseteq (a,b] = \bigcup_{i=1}^{\infty} (a_i, b_i] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i + \varepsilon)$$

$$\Rightarrow [a+\varepsilon, b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i + \varepsilon)$$

as $[a+\varepsilon, b]$ is compact in $\bar{\mathbb{R}}$

then

$\exists \{n_1, n_2, \ldots n_k\} = I$ s.t

$$[a+\varepsilon, b] \subseteq \bigcup_{i \in I} (a_i, b_i + \varepsilon) \subseteq \bigcup_{i \in I} (a_i, b_i + 2\varepsilon]$$

$$\Rightarrow (a+2\varepsilon, b] \subseteq [a+\varepsilon, b] \subseteq \bigcup_{i \in I} (a_i, b_i + 2\varepsilon] = \bigcup_{i \in I} I_i$$

as $(a+2\varepsilon, b] \subseteq \bigcup_{i \in I} (a_i, b_i + 2\varepsilon]$ (finite sets)

it is trivial to see that $(a+2\varepsilon, b]$ will be either completely inside some of the sets in $(a_i, b_i + 2\varepsilon]$ or will be inside some union of them

in first case (if it's completely inside some set)

$$\lambda((a+2\varepsilon, b]) \leq \lambda(I_j)$$

$$\leq \sum_{i \in I} \lambda(I_i)$$

or $\lambda((a+2\varepsilon, b]) \leq \sum_{i \in I'} \lambda(I_i) \leq \sum_{i \in I} \lambda(I_i)$

I' is all I_i sets where $(a+2\varepsilon, b]$ lies inside

as there will be overlap or touching, in both cases

$$\lambda((a+2\varepsilon, b]) \leq \sum_{i \in I'} \lambda(I_i)$$

so, we get $\lambda((a+2\varepsilon, b]) \leq \sum_{i \in I} (b_i + 2\varepsilon - a_i)$

$$\Rightarrow (b-a) - 2\varepsilon \leq \sum_{i \in I} (b_i - a_i) + \sum_{i \in I} 2\varepsilon$$

let $\epsilon \to 0$

$$\Rightarrow (b-a) \leq \sum_{i \in I} (b_i - a_i)$$

$$\Rightarrow (b-a) \leq \sum_{i=1}^{\infty} (b_i - a_i) \quad \text{——} (1)$$

now, case I : $-\infty < a, b < \infty$

then as $\{E_i\}_{i=1}^{\infty}$ s.t $E \in C$

then we can order $\{E_i\}_{i=1}^{\infty}$ s.t $E_i^\circ = (a_i, b_i]$

$a_{i+1}^\circ = b_i^\circ$

as if no such $a_i^\circ = b_i^\circ$ exist then

$\bigcup_{i=1}^{\infty} E_i$ will have a gap and ∞ not in C

then $\{E_i\}_{i=1}^{n}$ s.t $\bigcup_{i=1}^{n} E_i = (a_1, b_n] \in C$

now, $\lambda\left(\bigcup_{i=1}^{n} E_i\right) = b_n - a_1 = b_n - (a_n - b_n) - (a_{n-1} - b_{n-1}) \cdots - a_1$

$$= \sum_{i=1}^{n} (b_i^\circ - a_i^\circ)$$

then $b_n \leq b$

$a_1 = a$ ($\because$ ordering in such manner)

$$\Rightarrow \sum_{i=1}^{n} (b_i - a_i) \leq b - a$$

as all finite $a, b \Rightarrow a_i, b_i$ finite

$$\Rightarrow \lim_{n \to \infty} \sum_{i=1}^{n} (b_i - a_i) \leq b - a$$

$$\Rightarrow \sum_{i=1}^{\infty} (b_i - a_i) \leq b - a$$

from (1)

$$\sum_{i=1}^{\infty} (b_i - a_i) = b - a$$

case II : $a = -\infty$ or $b = \infty$

now

$b - a = \infty$

so in (1) : $\infty \leq \sum_{i=1}^{\infty} (b_i - a_i) \Rightarrow \infty = \sum_{i=1}^{\infty} (b_i - a_i)$

so, λ is a measure

Tutorial-2:

1. Non-negative linear combination of measure is also measure

$\lambda_1, \lambda_2 : \mathcal{C} \longrightarrow [0,\infty]$

s.t $\alpha\lambda_1 + \beta\lambda_2 : \mathcal{C} \longrightarrow [0,\infty]$

$(\Omega, \mathcal{F})$ $\mu : \mathcal{F} \longrightarrow [0,\infty]$

① $\mu(\phi) = 0$

② $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ s.t $E_i \cap E_j = \phi$ $i \neq j$

and $E = \bigcup_{i=1}^{\infty} E_i \in \mathcal{F}$

then $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$

let $\mu = \sum_{i=1}^{n} a_i \mu_i$ $\mu(\phi) = \sum_{i=1}^{n} a_i \mu_i(\phi) = 0$

$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{n} a_i \mu_i\left(\bigcup_{j\in\mathbb{N}} E_j\right)$

this step works even if $n=\infty$ but all values $\neq 0$ then only

$= \sum_{i=1}^{n} \sum_{j\in\mathbb{N}} a_i \mu_i(E_j)$

$= \sum_{j\in\mathbb{N}} \left(\sum_{i=1}^{n} a_i \mu_i(E_j)\right) = \sum_{j\in\mathbb{N}} \mu(E_j)$

2. let μ be measure on $(\Omega, \mathcal{F})$

$E \in \mathcal{F}$

$\mu_E : \mathcal{F} \longrightarrow [0,\infty]$

$\mu_E(\phi) = \mu(E \cap \phi) = \mu(\phi) = 0$

$\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ s.t $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ and $A_i \cap A_j = \phi$ $i \neq j$

then

$\mu_E\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(E \cap \bigcup_{i=1}^{\infty} A_i\right)$

$= \mu\left(\bigcup_{i=1}^{\infty} (E \cap A_i)\right)$

$i \neq j$

$(E \cap A_i) \cap (E \cap A_j)$

$= E \cap (A_i \cap A_j)$

$= E \cap (\phi)$

$= \phi$

$\mu\left(\bigcup_{i=1}^{\infty} (E \cap A_i)\right) = \sum_{i=1}^{\infty} \mu(E \cap A_i) = \sum_{i=1}^{\infty} \mu_E(A_i)$

3. Ω infinite set

$\mathcal{A} = \{E \subseteq \Omega \mid E \text{ is finite or } E^c \text{ is finite}\}$

$\mu : \mathcal{A} \longrightarrow [0,\infty)$ $\mu(E) = \begin{cases} 0 & ; E \text{ is finite} \\ 1 & ; E^c \text{ is finite} \end{cases}$

μ is finitely additive

$\{A_i\}_{i=1}^{n} \subseteq \mathcal{A}$, $A_i \cap A_j = \phi$ we have to show that

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$

case I: All A_i's are finite

$\bigcup_{i=1}^{n} A_i$ is finite

$$\Rightarrow \mu\left(\bigcup_{j=1}^{n} A_j\right) = 0 = \sum_{i=1}^{n} 0 = \sum_{j=1}^{n} \mu(A_j)$$

case II: one of the A_i not finite A_j is finite

$$A_j \cap A_i = \phi \ \forall\, i \neq j \Rightarrow A_i \subseteq A_j^c$$

$\Rightarrow$ All other A_i's are finite $i \neq j$

$\Rightarrow \mu\left(\bigcup_{j=1} A_j\right) = 1$ $\quad (\bigcup A_j)^c$ is finite

$$\sum_{i=1}^{n} \mu(A_i) = 1 = \mu\left(\bigcup_{i=1}^{n} A_i\right)$$

If Ω is countable infinite then μ is not σ-additive

let $A_j = \{\omega_j\}$ $\omega_j \in \Omega = \{\omega_1, \omega_2, \ldots\}$

$A_i \cap A_j = \phi$

$\bigcup_{i=1}^{\infty} A_i = \{\omega_1, \omega_2, \ldots\} = \Omega$

$\mu(\Omega) = 1 \neq \sum_{i=1}^{\infty} \mu(A_i) = 0$

if Ω is uncountable

case I: all A_j finite

$\bigcup_{j=1}^{\infty} A_j \in \mathcal{A} \Rightarrow \bigcup_{j=1}^{\infty} A_j$ - finite (or) uncountable ($\because$ def of $\mathcal{A}$)

↑

countable union of finite sets is countable so this cannot happen

$\Rightarrow \bigcup_{i=1}^{\infty} A_i$ is finite

So, $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 0 = \sum_{i=1}^{\infty} \mu(A_i)$

case II : one of the A_j not finite

$\Rightarrow A_j^c$ finite

$\Rightarrow A_i$ finite $(i \neq j)$ ($\because$ similar to before)

1. $\Omega \neq \phi$ $\phi \neq \mathcal{F} \subseteq \mathcal{P}(\Omega)$ σ-algebra

$\mu_1, \mu_2, \ldots \mu_n$ measures of $\mathcal{F}$

To prove: for $a_1, a_2, \ldots a_n \in [0, \infty)$, $\sum_{i=1}^{n} a_i \mu_i$ is a measure on $\mathcal{F}$

proof: let $\mu = \sum_{i=1}^{n} a_i \mu_i$ then

$$\mu(\phi) = \sum_{i=1}^{n} a_i \mu_i(\phi) = \sum_{i=1}^{n} 0 = 0 \quad (\because \mu_i(\phi) = 0 \ \forall i \text{ as } \mu_i \text{ is a measure}) \quad \text{---} ①$$

now let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ s.t $E_i \cap E_j = \phi$ for $i \neq j$ and $E = \bigcup_{i=1}^{\infty} E_i \in \mathcal{F}$

then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu(E) = \sum_{i=1}^{n} a_i \mu_i(E)$$

$$= \sum_{i=1}^{n} a_i \sum_{j=1}^{\infty} \mu_i(E_j)$$

$$= \sum_{j=1}^{\infty} \sum_{i=1}^{n} a_i \mu_i(E_j) \quad \left(\because \text{all } \geq 0, \text{ so we can interchange } \textstyle\sum \text{ freely}\right)$$

$$= \sum_{j=1}^{\infty} \left(\mu(E_j)\right) \quad \text{---} ②$$

so, from ①, ②, μ is a measure

2. $\Omega \neq \phi$ $\phi \neq \mathcal{F} \subseteq \mathcal{P}(\Omega)$ σ-algebra $E \in \mathcal{F}$

$\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{F}$, μ is a measure of $\mathcal{F}$

$$\mu_E(\phi) = \mu(\phi \cap E) = \mu(\phi) = 0 \quad (\because \mu \text{ is a measure}) \quad \text{---} ①$$

and let $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ s.t $A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ and $A_i \cap A_j = \phi$

then $\mu_E\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right)$

$$= \mu\left(\bigcup_{i=1}^{\infty} (E \cap A_i)\right)$$

as $A_i \cap A_j = \phi$ for $i \neq j$

$(E \cap A_i) \cap (E \cap A_j) = E \cap \phi = \phi$ for $i \neq j$

$$\Rightarrow \mu_E\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(E \cap A_i) = \sum_{i=1}^{\infty} \mu_E(A_i) \quad \text{---} ②$$

from ①, ②, μ_E is a measure on $\mathcal{F}$

3. $\Omega \leftarrow$ infinite set $\quad \mathcal{A} = \{E \subseteq \Omega \mid E \text{ or } X \setminus E \text{ is finite}\}$

$$\mu : \mathcal{A} \longrightarrow [0, \infty)$$

$$\mu(E) = \begin{cases} 0 ; & E \text{ is finite} \\ 1 ; & E^c \text{ is finite} \end{cases}$$

To prove : μ is finitely additive

proof :

$\phi \in \mathcal{A}$ as $|\phi| = 0$ or ϕ is finite

$\Rightarrow \mu(\phi) = 0$ as ϕ is finite —— (1)

and now let $\{E_i\}_{i=1}^{n} \subseteq \mathcal{A}$ be s.t for $i \neq j$, $E_i \cap E_j = \phi$

and $E = \bigcup_{i=1}^{n} E_i \in \mathcal{A}$, then

case I : all E_i are finite

in this case

$\bigcup_{j=1}^{n} E_j$ is finite and so,

$$\mu\left(\bigcup_{j=1}^{n} E_j\right) = 0 \text{ and } \mu(E_i) = 0 \quad \forall i \in \{1, \ldots n\}$$

$$\Rightarrow \mu(E) = 0 = \sum_{i=1}^{n} \mu(E_i)$$

case II : $\exists j \in \{1, 2 \ldots n\}$ s.t E_j is not finite

then as $E_i \cap E_j = \phi$ for $i \neq j$

$\Rightarrow E_i \subseteq E_j^c \ \forall \ i \neq j$ but as $E_j \in \mathcal{A}$,

E_j or E_j^c has to be finite

$\Rightarrow E_j^c$ is finite

$\Rightarrow E_i \subseteq E_j^c$ is finite $\forall i \neq j$

so, $\mu(E_i) = \begin{cases} 0 ; & i \neq j \\ 1 ; & i = j \end{cases}$

$\Rightarrow \sum_{i=1}^{n} \mu(E_i) = 1 = \mu(E)$ as E is infinite

($\because$ union of one infinite set)

so, μ is finitely additive

now as Ω is infinite

case I : Ω is countable infinite

here let $\Omega = \{\omega_1, \omega_2, \ldots\}$

let $A_i = \{\omega_i\}$ then

$\mu(A_i) = 0 \quad \forall i \in \mathbb{N}$, but

$\bigcup_{i=1}^{\infty} A_i = \Omega$ and $\mu(\Omega) = 1 \neq \sum_{i=1}^{\infty} \mu(A_i)$

so if Ω is countable infinite, then μ is not σ-additive

Case II: Ω is uncountable, $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ s.t $A = \bigcup_{i=1}^{\infty} A_i$, $A_i \cap A_j = \phi$ $\forall i \neq j$
$A \in \mathcal{A}$

Here if all A_i finite then

as $A \in \mathcal{A}$, A is either finite (or) uncountable
($\because$ definition of $\mathcal{A}$, as if A countable then A also infinite)

but $A = \bigcup_{i=1}^{\infty} A_i$ is countable union of finite sets

$\Rightarrow A$ is countable

$\Rightarrow A$ is finite ($\because$ A cannot be countable infinite from above)

$\Rightarrow \mu(A) = 0 = \sum_{i=1}^{\infty} \mu(A_i)$

if $\exists j \in \mathbb{N}$ s.t A_j is uncountable ($\because$ cannot be countable infinite from above)
as $\forall i \neq j$ $A_i \cap A_j = \phi$

$\Rightarrow A_i \subseteq A_j^c$ finite

$\Rightarrow A_i$ finite $\forall i \in \mathbb{N} \setminus \{j\}$

so, $\sum_{i=1}^{\infty} \mu(A_i) = \mu(A_j) = 1 = \mu(A)$
(A: infinite)

so, if Ω is uncountable then μ is σ-additive

Tutorial-3:

Carathéodory extension:

given $\mathcal{A} \subseteq P(\Omega)$ is an algebra and $\nu : \mathcal{A} \longrightarrow [0,\infty]$ measure

$\pi^* : P(\Omega) \longrightarrow [0,\infty]$ as

$$\pi^*(E) = \inf\left\{ \sum_{j=1}^{\infty} \nu(E_i) \;\middle|\; E \subseteq \bigcup_{j=1}^{\infty} E_j,\ E_j \in \mathcal{A} \right\}$$

outermeasure i.e $\pi^*(\phi) = 0$

$E \subseteq \bigcup_j E_j$ $E_j \subseteq \Omega$ then $\pi^*(E) \leq \sum_{i=1}^{\infty} \pi^*(E_i)$

$$\mathcal{M} = \{ A \in P(\Omega) \mid \pi^*(E) = \pi^*(E \cap A) + \pi^*(E \cap A^c)\ \forall E \in P(\Omega) \}$$

(A is π^*-measurable)

$\mathcal{M}$ is an algebra and $\mathcal{A} \subset \mathcal{M}$

$\mathcal{M}$ is σ-algebra then $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{M}$ and if $\pi^*|_{\mathcal{A}} = \nu$ then

$\pi^*(A) = \nu(A)$ $\forall A \in \mathcal{A}$

1. $\{F_i\}_{i=1}^{n} \subseteq \mathcal{M}$ disjoint collection then $\forall E \subseteq \Omega$

$$\pi^*(E \cap (\bigcup_{i=1}^{n} F_i)) = \sum_{j=1}^{n} \pi^*(E \cap F_j)$$

then π^* is finitely additive by taking $E \subseteq \Omega$

2. $\mathcal{M}$ is σ-algebra as

$\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ then $A = \bigcup_{j=1}^{\infty} E_j$

show monotone by $E \subseteq \bigcup E_j$, $E_j := E$ else ϕ

if A is π^* measurable, then we are done

$$\bigcup_{j=1}^{\infty} E_j \in \mathcal{M} \Leftrightarrow \pi^*(E \cap (\bigcup_{j=1}^{\infty} E_j)) + \pi^*(E \cap (\bigcup_{j=1}^{\infty} E_j)^c) = \pi^*(E)\ \forall E \subseteq \Omega$$

(monotonicity)

$\Rightarrow$ comes from definition

$\Leftarrow$ if we show this inclusion we are done

now, enough to show $\pi^*(E) \geq \pi^*(E \cap (\bigcup_{j=1}^{\infty} E_j)) + \pi^*(E \cap (\bigcup_{j=1}^{\infty} E_j)^c)$

$$E \cap (\bigcup_{i=1}^{\infty} E_i)^c \subseteq E \cap (\bigcup_{i=1}^{n} E_i)^c$$

$$\Rightarrow \pi^*(E \cap (\bigcup_{i=1}^{\infty} E_i)^c) \leq \pi^*(E \cap (\bigcup_{i=1}^{n} E_i)^c) \quad (\because \pi^* \text{ is monotone})$$

as $\mathcal{M}$ is an algebra

$$\pi^*(E) = \pi^*(E \cap (\bigcup_{i=1}^{n} E_i)) + \pi^*(E \cap (\bigcup_{i=1}^{n} E_i)^c)$$

$$\pi^*(E) \geq \pi^*(E\cap(\bigcup_{j=1}^{n}E_j)) + \pi^*(E\cap(\bigcup_{j=1}^{\infty}E_j)^c)$$

now let $F_1=E_1$, $F_2=E_2\setminus E_1$, $\ldots$ $F_n=E_n\setminus\bigcup_{j=1}^{n-1}E_j \in M$

($\because M$ is an algebra)

$$\bigcup_{i=1}^{n}F_i = \bigcup_{j=1}^{n}E_j$$

$$\Rightarrow \mu^*(E\cap(\bigcup_{i=1}^{n}F_i)) = \sum_{i=1}^{n}\mu^*(E\cap F_i)$$

$$\pi^*(E\cap(\bigcup_{i=1}^{n}E_i)) = \sum_{i=1}^{n}\mu^*(E\cap(E_i\setminus\bigcup_{j=1}^{i-1}E_j))$$

$$\Rightarrow \pi^*(E) \geq \sum_{j=1}^{n}\pi^*(E\cap F_j) + \pi^*(E\cap(\bigcup_{j=1}^{\infty}E_j)^c)$$

then as $n\to\infty$

$$\pi^*(E) \geq \sum_{j=1}^{\infty}\pi^*(E\cap F_j) + \pi^*(E\cap(\bigcup_{j=1}^{\infty}E_j)^c)$$

$\{F_j\}_{j=1}^{\infty}\subseteq M \to$ disjoint collection

$$\Rightarrow \pi^*(E\cap(\bigcup_{j=1}^{\infty}F_j)) \leq \sum_{j=1}^{\infty}\pi^*(E\cap F_j) \quad (\because \text{subadditivity})$$

$$\Rightarrow \pi^*(E\cap(\bigcup_{j=1}^{\infty}E_j)) \leq \sum_{j=1}^{\infty}\pi^*(E\cap F_j)$$

$$\Rightarrow \pi^*(E) \geq \pi^*(E\cap(\bigcup_{j=1}^{\infty}E_j)) + \pi^*(E\cap(\bigcup_{j=1}^{\infty}E_j)^c)$$

3. $\pi^*|_{\mathcal{A}} = \nu$ $\quad \nu(A) \geq \pi^*(A)$ ($\because$ definition of π^*)

$A\subseteq A\cup\phi\cup\phi\cdots$

now, $\pi^*(A) = \inf\{$ $\}$

$E_j\in\mathcal{A}$ s.t $A\subseteq\bigcup E_j$ then $\nu(A)\leq\sum_{j=1}^{\infty}\nu(E_j)$

take inf $\Rightarrow \nu(A)\leq\pi^*(A)$

1. $\{F_i\}_{i=1}^{n}\subseteq M$ is disjoint collection $\forall E\subseteq\Omega$

$n=1$: $F_1\in M$, $E\subseteq\Omega$ then trivial

if true for $n=k$, then for

$n=k+1$: $\pi^*(E\cap(\bigcup_{i=1}^{k+1}F_i)) = \pi^*(E\cap(\bigcup_{i=1}^{k+1}F_i)\cap F_{k+1}) + \pi^*(E\cap(\bigcup_{i=1}^{k+1}F_i)\cap F_{k+1}^c)$

$(\bigcup_{i=1}^{k+1}F_i)\cap F_{k+1} = F_{k+1}$ $\quad (\bigcup_{i=1}^{k+1}F_i)\cap F_{k+1}^c = \bigcup_{i=1}^{k}F_i$

$$\Rightarrow \pi^*(E\cap(\overset{k+1}{\underset{i=1}{\cup}}F_i)) = \pi^*(E\cap F_{k+1}) + \pi^*(E\cap(\overset{k}{\underset{i=1}{\cup}}F_i))$$
$$= \sum_{i=1}^{k+1}\pi^*(E\cap F_i)$$

so, by induction true

2. To prove: M is σ-algebra

proof:

$\{E_j\}_{j=1}^{\infty} \subseteq M$ then $A = \overset{\infty}{\underset{j=1}{\cup}}E_j$

if A is π^* measurable then we are done

$$\overset{\infty}{\underset{j=1}{\cup}}E_j \in M \Leftrightarrow \pi^*(E\cap(\overset{\infty}{\underset{j=1}{\cup}}E_j)) + \pi^*(E\cap(\overset{\infty}{\underset{j=1}{\cup}}E_j)^c) = \pi^*(E) \quad \forall E\subseteq\Omega$$

$\geq$ comes from definition

$\leq$ $\rightarrow$ we will show

now, if $E\subseteq F, E,F\subseteq\Omega$ then

$$\pi^*(E) = \inf_{E\subseteq F}\left\{\sum_{i=1}^{\infty}\nu(E_i) \mid A\subseteq\overset{\infty}{\underset{i=1}{\cup}}E_i, E_i\in\mathcal{A}\right\}$$

$\forall\varepsilon>0, \exists \{F_i\}_{i=1}^{\infty}\subseteq\mathcal{A}$ s.t $\sum_{i=1}^{\infty}\nu(F_i)\leq\pi^*(F)+\varepsilon$

$$\overset{\infty}{\underset{i=1}{\cup}}F_i \supseteq E \Rightarrow \pi^*(E)\leq\sum_{i=1}^{\infty}\nu(F_i)\leq\pi^*(F)+\varepsilon$$

as $\varepsilon\to 0$

$$\Rightarrow \pi^*(E)\leq\pi^*(F)$$

now, $E\cap(\overset{\infty}{\underset{i=1}{\cup}}E_i)^c \subseteq E\cap(\overset{n}{\underset{i=1}{\cup}}E_i)^c$

$$\Rightarrow \pi^*(E\cap(\overset{\infty}{\underset{i=1}{\cup}}E_i)^c)\leq\pi^*(E\cap(\overset{n}{\underset{i=1}{\cup}}E_i)^c)$$

also, $\pi^*(E) = \pi^*(E\cap(\overset{n}{\underset{i=1}{\cup}}E_i)) + \pi^*(E\cap(\overset{n}{\underset{i=1}{\cup}}E_i)^c)$

$$\pi^*(E)\geq\pi^*(E\cap(\overset{n}{\underset{i=1}{\cup}}E_i)) + \pi^*(E\cap(\overset{\infty}{\underset{i=1}{\cup}}E_i)^c)$$

let $F_1 = E_1$, $F_2 = E_2\setminus E_1$, $F_3 = E_3\setminus(E_1\cup E_2)$, $\ldots$ $F_n = E_n\setminus(\overset{n-1}{\underset{i=1}{\cup}}E_i)\in M$ ($\because$ M is algebra)

$$\overset{n}{\underset{i=1}{\cup}}E_i = \overset{n}{\underset{i=1}{\cup}}F_i$$

$$\Rightarrow \pi^*(E\cap(\overset{n}{\underset{i=1}{\cup}}F_i)) = \sum_{i=1}^{n}\pi^*(E\cap F_i)$$

$$\pi^*(E\cap(\overset{n}{\underset{i=1}{\cup}}E_i)) = \sum_{i=1}^{n}\pi^*(E\cap(E_i\setminus\overset{i-1}{\underset{j=1}{\cup}}E_j))$$

$$\pi^*(E)\geq\pi^*(E\cap(\overset{n}{\underset{i=1}{\cup}}E_i)) + \pi^*(E\cap(\overset{\infty}{\underset{i=1}{\cup}}E_i)^c)$$

$$\sum_{j=1}^{n}\pi^*(E\cap F_j)$$

as $n\to\infty$

$$\Rightarrow \pi^*(E)\geq\sum_{j=1}^{\infty}\pi^*(E\cap F_j) + \pi^*(E\cap(\overset{\infty}{\underset{i=1}{\cup}}E_i)^c)$$

$\{F_j\}_{j=1}^{\infty}\rightarrow$ disjoint

$$\pi^*(E\cap(\overset{\infty}{\underset{j=1}{\cup}}F_j))\leq\sum_{j=1}^{\infty}\pi^*(E\cap F_j)$$ ($\because$ subadditivity) done in class

$$\Rightarrow \pi^*(E)\geq\pi^*(E\cap(\overset{\infty}{\underset{j=1}{\cup}}E_j)) + \pi^*(E\cap(\overset{\infty}{\underset{i=1}{\cup}}E_i)^c)$$

3. To prove: $\pi^*|_{\mathcal{A}} = \nu$, $\pi^*(A) = \nu(A)$ $\forall A \in \mathcal{A}$

proof: $\pi^*(A) \leq \nu(A)$
($\because$ definition)

now for $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ s.t. $A \subseteq \bigcup_{i=1}^{\infty} E_i$

then $\nu(A) \leq \nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu(E_i)$

($\because \nu$ is σ-measure)

$\Rightarrow \nu(A) \leq \inf\{\quad\}$

$\Rightarrow \nu(A) \leq \pi^*(A)$

so, $\nu(A) = \pi^*(A)$

Tutorial-4:

$S := \{(a,b] \mid -\infty < a < b < \infty\} \cup \{\phi\} \subset \mathcal{P}(\mathbb{R})$

$\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ $F: \mathbb{R} \to \mathbb{R}$ non-decreasing right continuous

$F: \bar{\mathbb{R}} \to \bar{\mathbb{R}}$

$F(-\infty) = \inf\{F(x) \mid x \in \mathbb{R}\}$
$F(\infty) = \sup\{F(x) \mid x \in \mathbb{R}\}$

$\lambda_F: S \to [0,\infty]$

$$\lambda_F(S) = \begin{cases} 0 & ; S = \phi \\ F(b) - F(a) & ; S = (a,b] \end{cases}$$

1. $\{(a_i,b_i] \mid i = 1,2\ldots K\} \subseteq S$ disjoint $(a_i,b_i] \subseteq (a,b]$

$a_i, b_i, a, b \in \bar{\mathbb{R}} \quad \forall i = 1,2\ldots K$

To prove: $\sum_{i=1}^{K} \lambda_F((a_i,b_i]) \leq \lambda_F((a,b])$

proof:

if $\lambda_F((a,b]) = \infty$ then this follows trivially

if $\lambda_F((a,b]) < \infty$ then $F(a), F(b) \in \mathbb{R}$

$(a_i,b_i] \subseteq (a,b]$

i.e

$a \leq a_1 \leq b_1 \leq a_2 \leq b_2 \ldots \leq a_{n+1} \leq b_{n+1} \leq a_n \leq b_n \leq b$

$\Rightarrow F(a_i) < \infty \quad F(b_i) < \infty$ ($\because F$ is non-decreasing)
$F(b_i) < \infty$

also, $F(a) \leq F(a_1) \leq \ldots \leq F(b_{n-1}) \leq F(a_n) \leq F(b_n) \leq F(b)$
($\because F$ is non-dec)

$\Rightarrow F(b) - F(a) = F(b) - F(b_n) + F(b_n) - F(a_n) + F(a_n) \ldots - F(a_1) + F(a_1) - F(a)$

$\geq \sum_{n=1}^{n} F(b_n) - F(a_n)$

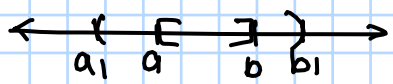
$\Rightarrow \lambda_F((a,b]) \geq \sum_{i=1}^{n} \lambda_F((a_i,b_i])$

2. $(a,b] \subseteq \bigcup_{i=1}^{K} (a_i,b_i) \quad a_i < b_i \quad a,b,a_i,b_i \in \mathbb{R} \quad \forall i = 1,2,\ldots K$

To prove: $\sum_{i=1}^{K} F(b_i) - F(a_i) \geq F(b) - F(a)$

proof: now, for $K = 1$:

$(a,b] \subseteq (a_1,b_1)$



$F(a_1) \leq F(a) \leq F(b) \leq F(b_1)$

$$F(b)-F(a_1) \le F(b_1)-F(a_1)$$

$$\Rightarrow \quad F(b)-F(a) \le F(b)-F(a_1) \le F(b_1)-F(a_1)$$

$$\Rightarrow F(b)-F(a) \le F(b_1)-F(a_1)$$

if true for $k=n$ then for $k=n+1$:

$$\hookrightarrow \sum_{i=1}^{n} F(b_i)-F(a_i) \ge F(b)-F(a)$$

for $[a,b] \subseteq \bigcup_{i=1}^{n}(a_i,b_i)$

now let $p \in \{1,2,..n+1\}$ s.t $b_p = \max_{1\le j\le n+1} b_j$ p is the index which covers

case I: $a_p \le a$: then

$$F(b_p)-F(a_p) \le \sum_{j=1}^{n+1} F(b_j)-F(a_j)$$

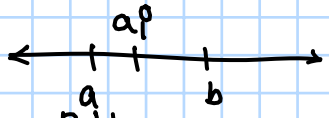
$(\because$ trivial$)$

$$\Rightarrow F(b)-F(a) \le F(b_i)-F(a_i) \le \sum_{j=1}^{n+1} F(b_j)-F(a_j)$$

$(\because k=1$ case$)$

case II: $a_p \ge b$ then this follows trivially

case III: $a < a_p < b$ then



$$[a,a_i] \subseteq \bigcup_{i=1}^{n+1}(a_p,b_p)$$

but $[a,a_p] \cap (a_i,b_i) = \phi$

$$\Rightarrow [a,a_i] \subseteq \bigcup_{\substack{j=1 \\ j\neq p}}^{n+1}(a_j,b_j)$$

$$\Rightarrow F(a_i)-F(a) \le \sum_{\substack{j=1 \\ j\neq p}}^{n+1} F(a_j)-F(b_j)$$

$$\Rightarrow F(b_p)-\cancel{F(a_i)}-F(a)+\cancel{F(a_i)} \le \sum_{j=1}^{n+1} F(a_j)-F(b_j)$$

$$\Rightarrow F(b)-F(a) \le \sum_{j=1}^{n+1} F(a_j)-F(b_j)$$

3. $\mathcal{B}(\mathbb{R})$ $\mu: \mathcal{B}(\mathbb{R}) \longrightarrow [0,\infty]$

↳ additive $\mu(I)<\infty$ $\forall$ bounded $I \subset \mathbb{R}$

To prove: $\exists$ non-dec fution F $\mu((a,b]) = \lambda_F((a,b])$ $\forall a,b \in \mathbb{R}$ and μ measure then F right cont

proof:

$$\mu((a,b]) = \lambda_F((a,b])$$

$$F(x) = \begin{cases} \mu((0,x]) ; & x>0 \\ 0 ; & x=0 \\ -\mu((x,0]) ; & x<0 \end{cases}$$

not unique as scaled

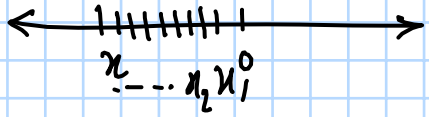
trivial to see it's non decreasing by monotonity of μ

if $x \neq 0$: $0 \leq x \leq \dots \leq x_n \dots \leq x_2 \leq x_1$

$$F(x_1) = \mu((0,x_1]) = \mu((0,x]) \cup ((x,x_1])$$
$$= \mu((0,x]) + \mu((x,x_1])$$
$$= F(x) + \sum_{n=1}^{\infty} \mu((x_{n+1}, x_n])$$
$$= F(x) + \lim_{K\to\infty} F(x_1) - F(x_{K+1})$$
$$\Rightarrow \lim_{K\to\infty} F(x_{K+1}) = F(x)$$

case II: $x<0$

$$(x,0] = (x,x_1] \cup (x_1,0]$$
$$\mu((x,0]) = \mu((x,x_1]) + \mu((x_1,0])$$
$$\Rightarrow -F(x) = -F(x_1) + \underbrace{\mu\left(\bigcup_{n=1}^{\infty} (x_{n+1},x_n]\right)}_{= \lim_{K\to\infty} \mu((x_K, x_1])}$$
$$= \mu(x_1) - \lim_{K\to\infty} F(x_K)$$



so, we are done

1. $X \neq \phi$ $\mathcal{C} = \{E \subseteq X \mid E \text{ or } E^c \text{ finite}\}$

Claim: $\mathcal{C}$ is an algebra, X is finite then $\mathcal{P}(X) = \mathcal{C}$

proof:

① $\phi \in \mathcal{C}$ as ϕ is finite

$X^c = \phi$ finite $\Rightarrow X \in \mathcal{C}$

② for $A \in \mathcal{C}$, $B \in \mathcal{C}$ we have
A is finite or A^c is finite
and same for B

then $A \cap B$ is s.t if infinite, then A^c, B^c finite
$\Rightarrow A^c \cup B^c$ finite
$\Rightarrow A \cap B \in \mathcal{C}$

if finite, then $\Rightarrow A \cap B \in \mathcal{C}$

③ $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$ is trivial

from ①, ②, ③ $\mathcal{C}$ is an algebra

also if X is finite then

$\forall A \subseteq X \Rightarrow A$ is finite

$\Rightarrow \forall A \in \mathcal{P}(X) \Rightarrow A \in \mathcal{C}$

$\Rightarrow \mathcal{P}(X) \subseteq \mathcal{C}$

as $\mathcal{C} \subseteq \mathcal{P}(X) \Rightarrow \mathcal{P}(X) = \mathcal{C}$

2. $X = \{1,2,3,4\}$

$S_1 = \{\phi, X, \{1\}, \{2\}, \{3,4\}\}$

$S_2 = \{\phi, X, \{1\}, \{3\}, \{2,4\}\}$

$S_1 \cap S_2 = \{\phi, X, \{1\}\}$

$\{1\}^c = \{2,3,4\} \rightarrow$ Not union of finite

$\Rightarrow S_1 \cap S_2$ not semi-algebra

3. $X, Y \neq \phi$

S_1, S_2 semi-algebra

$S_1 \subseteq \mathcal{P}(X) \quad S_2 \subseteq \mathcal{P}(Y)$

$S_1 \times S_2 = \{F \times E \mid F \in S_1, E \in S_2\}$

To prove: $S_1 \times S_2 \subseteq \mathcal{P}(X \times Y)$ is semi-algebra

proof: $X \in S_1, Y \in S_2 \Rightarrow X \times Y \in S_1 \times S_2$ — ①

now, for $A = A_1 \times A_2 \in S_1 \times S_2$

$B = B_1 \times B_2 \in S_1 \times S_2$

$A \cap B = (A_1 \cap B_1) \times (A_2 \cap B_2)$

as $A_1, B_1 \in S_1 \Rightarrow A_1 \cap B_1 \in S_1$

$A_2, B_2 \in S_2 \Rightarrow A_2 \cap B_2 \in S_2$

so, $A \cap B \in S_1 \times S_2$ — ②

also, $\forall A \in S_1 \times S_2$ then $A = A_1 \times A_2$

now $\exists \{E_{i,1}\}_{i=1}^{n} \subseteq S_1$ disjoint

$\{E_{i,2}\}_{i=1}^{m} \subseteq S_2$ disjoint

s.t $A_1^c = \bigsqcup_{i=1}^{n} E_{i,1}^{\circ}$

$A_2^c = \bigsqcup_{i=1}^{m} E_{i,2}^{\circ}$

then wlog $n=m$ (as we add ϕ when $n<m$ or $n>m$)

let $E_i^{\circ} = E_{i,1}^{\circ} \times E_{i,2}^{\circ} \in S_1 \times S_2$

then $E_i^{\circ} \cap E_j^{\circ} = \phi \quad \forall i \neq j$

also, $A^c = (A_1^c \times A_2^c) \cup (A_1^c \times A_2) \cup (A_1 \times A_2^c)$

$= \bigsqcup_{i=1}^{n} (E_{i,1}^{\circ} \times E_{i,2}^{\circ}) \cup (A_1^c \times A_2) \cup (A_1 \times A_2^c)$ — ③

(disjoint; disjoint; disjoint; complements)

so, from ①, ②, ③ $S_1 \times S_2$ is a semi-algebra

4. $X \neq \phi \quad \phi \neq E \subseteq X \quad \mathcal{C}, \mathcal{A}$

$\mathcal{C} \cap E := \{A \cap E \mid A \in \mathcal{C}\}$

$\mathcal{A} \cap E := \{A \cap E \mid A \in \mathcal{A}\}$

now $E \in \mathcal{C} \cap E$ as $X \in \mathcal{C}$, $X \cap E = E$

$E \in \mathcal{A} \cap E$ as $X \in \mathcal{A}$, $X \cap E = E$ — ①

also, for $\tilde{A}, \tilde{B} \in \mathcal{C} \cap E$

$\exists\, A \in \mathcal{C}$ s.t $\tilde{A} = A \cap E$

$B \in \mathcal{C}$ s.t $\tilde{B} = B \cap E$

then $\tilde{A} \cap \tilde{B} = A \cap B \cap E$

$= (A \cap B) \cap E$

$\in \mathcal{C}$

$\Rightarrow \tilde{A} \cap \tilde{B} \in \mathcal{C}$

same for $\mathcal{A}$ — ②

and now $\forall \tilde{A} \in \mathcal{C} \cap E$, $\exists A \in \mathcal{C}$ s.t

$\tilde{A} = A \cap E$

$\exists \{E_i^{\circ}\}_{i=1}^{m}$ disjoint

s.t $A = \bigsqcup_{i=1}^{m} E_i^{\circ}$

$\Rightarrow A \cap E = \bigsqcup_{i=1}^{m} (E_i \cap E)$

$\{E_i \cap E\}_{i=1}^{m}$ → disjoint in $\mathcal{C} \cap E$

also $\tilde{A} \in \mathcal{A} \cap E \Rightarrow \exists A \in \mathcal{A}$ s.t $\tilde{A} = A \cap E$



$E \setminus \tilde{A} = E \cap (\tilde{A})^c$

$= E \cap (A \cap E)^c$

$= E \cap (A^c \cup E^c)$

$= E \cap A^c$ as $E \cap E^c = \phi$

so, $E \setminus \tilde{A} \in \mathcal{A} \cap E$ — ③

from ①, ②, ③ we have $\mathcal{C} \cap E$ to be semi-algebra
$\mathcal{A} \cap E$ to be algebra

5. $X, Y \neq \phi$ $f: X \to Y$
$E \subseteq Y$

$$f^{-1}(E) := \{x \in X \mid f(x) \in E\}$$

$\mathcal{C}$ semialgebra in $\mathcal{P}(Y)$

$$\mathcal{C} \subseteq \mathcal{P}(Y)$$

$$f^{-1}(\mathcal{C}) = \{f^{-1}(E) \mid E \in \mathcal{C}\}$$

$$= \{\{x \in X \mid f(x) \in E\} \mid E \in \mathcal{C}\}$$

as $Y \in \mathcal{C} \Rightarrow f: X \to Y$
so, $f^{-1}(Y) = \{x \in X \mid f(x) \in Y\}$

$$= X \text{ as } f(X) \subseteq Y$$

$$\Rightarrow X \in f^{-1}(\mathcal{C}) \quad \text{—①}$$

now, $\tilde{A}, \tilde{B} \in f^{-1}(\mathcal{C})$ i.e $\exists A, B \in \mathcal{C}$ s.t

$$\tilde{A} = \{x \in X \mid f(x) \in A\}$$

$$\tilde{B} = \{x \in X \mid f(x) \in B\}$$

also $A \cap B \in \mathcal{C} \Rightarrow \{x \in X \mid f(x) \in A \cap B\} \in f^{-1}(\mathcal{C})$

now, $\forall x \in \tilde{A} \cap \tilde{B}$, $f(x) \in A$, $f(x) \in B$

$$\Rightarrow f(x) \in A \cap B$$

$$\Rightarrow x \in \{x \in X \mid f(x) \in A \cap B\}$$

$$\Rightarrow \tilde{A} \cap \tilde{B} \subseteq \{x \in X \mid f(x) \in A \cap B\}$$

also, $\forall x \in \{x \in X \mid f(x) \in A \cap B\}$
we have
$f(x) \in A$, $f(x) \in B$

$$\Rightarrow x \in \{x \in X \mid f(x) \in A\} = \tilde{A}$$
$$x \in \{x \in X \mid f(x) \in B\} = \tilde{B}$$

$$\Rightarrow x \in \tilde{A} \times \tilde{B}$$

so, $\{x \in X \mid f(x) \in A \cap B\} = \tilde{A} \times \tilde{B}$

so, $\tilde{A} \cap \tilde{B} \in f^{-1}(\mathcal{C})$ —②

now, let $\tilde{A} \in f^{-1}(\mathcal{C})$ then $\exists A \in \mathcal{C}$ s.t

$$\tilde{A} = \{x \in X \mid f(x) \in A\}$$

$\exists \{E_i\}_{i=1}^{n} \subseteq \mathcal{C}$ s.t $E_i \cap E_j = \phi$ $i \neq j$ and $A = \bigcup_{i=1}^{n} E_i$

$$\tilde{A} = \{x \in X \mid f(x) \in \bigsqcup_{i=1}^{n} E_i\}$$

now, $\{x \in X \mid f(x) \in E \cup F\} = \{x \in X \mid f(x) \in E\} \cup \{x \in X \mid f(x) \in F\}$

A B C

$E \cup F$: $\uparrow$ disjoint

$B \cup C$: disjoint

as $\forall x \in A$, $f(x) \in E$ or $f(x) \in F$

$\Rightarrow x \in B$ or $x \in C$

if $x \in B$ or $x \in C \Rightarrow x \in A$ is trivial

now if $E \cap F = \phi$ then

if $B \cap C \neq \phi$

$\exists x$ in both

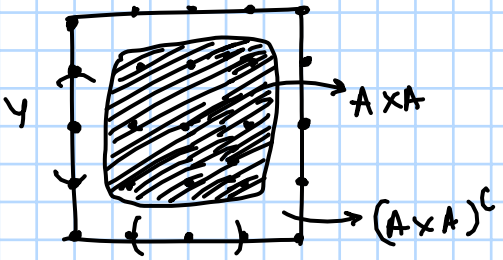
but then $x \in E$ & $x \in F$ contradiction

so, $\tilde{A} = \{.. E_i ..\} \cup \{.. E_j ..\}$ — ③

$\uparrow$ disjoint

from ①, ②, ③ $f^{-1}(\mathcal{E})$ is semi-algebra

6. $\mathcal{A}_1 \subseteq \mathcal{P}(X)$, $\mathcal{A}_2 \subseteq \mathcal{P}(Y)$



$X = \{0,1,2,3,4,5\} = Y$

$\mathcal{A}_1 = \{\phi, X, \{2,3,4\}, \{1,3\}\} = \mathcal{A}_2$

$\{2,3,4\}$: A

but $(A \times A)^c$ not in $\mathcal{A}_1 \times \mathcal{A}_2$

7. $\Omega = \{1,2,3,4\}$

$S = \{\{1\}, \{2,3\}\} \subseteq \mathcal{P}(\Omega)$

$\mathcal{F}(S) = \{\phi, \Omega, \{1\}, \{2,3\}, \{1,2,3\}, \{2,3,4\}, \{4\}, \{1,4\}\}$

let μ, ν be s.t

$\mu(\{1\}) = 1 \quad \mu(\{2,3\}) = 5$

$\nu(\{1\}) = 1 \quad \nu(\{2,3\}) = 5$

satisfies all properties

	ϕ	Ω	$\{1\}$	$\{4\}$	$\{1,4\}$	$\{2,3\}$	$\{1,2,3\}$	$\{4,3,4\}$
μ	0	10	1	4	5	5	6	9
ν	0	11	1	5	6	5	6	10

$\uparrow$ both satisfy properties of sigma-algebra

8. $(\Omega, \mathcal{F}, \mu)$ → measure space

$\forall E, F \in \mathcal{F}$ $\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$

if $\mu(E \cup F) < \infty \Rightarrow \mu(E) < \infty, \mu(F) < \infty$ (∵ monotonicity)



$E \cup F = \underbrace{(E \cap F^c) \cup (E^c \cap F) \cup (E \cap F)}_{\text{all 3 disjoint}}$

$\Rightarrow \mu(E \cup F) = \mu(E \cap F) + \mu(E^c \cap F) + \mu(E \cap F^c)$

now, $E = \underbrace{(E \cap F^c) \cup (E \cap F)}_{\text{both disjoint}}$

$F = \underbrace{(F \cap E^c) \cup (F \cap E)}_{\text{Both disjoint}}$

so, $\mu(F) = \mu(F \cap E^c) + \mu(F \cap E)$

$\mu(E) = \mu(E \cap F) + \mu(E \cap F^c)$

$\Rightarrow \mu(E) + \mu(F) = \mu(F \cap E^c) + \mu(E \cap F^c) + \mu(E \cap F) + \mu(E \cap F)$

$= \mu(E \cup F) + \mu(E \cap F)$

if $\mu(E \cup F) = \infty$

then $\mu(E \cup F) = \underbrace{\mu(E \cap F^c) + \mu(E^c \cap F) + \mu(E \cap F)}_{\text{one } \infty}$

$\Rightarrow \mu(E) = \infty$ or $\mu(F) = \infty$ (∵ monotonicity)

so above trivially true

9. $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$ is $\mathcal{F}(\underbrace{\quad}_{\text{open sets in } \mathbb{R}})$

(a) $S_1 = \{(a,b) \mid a < b\}$

then $(a,b) \in \mathcal{B}(\mathbb{R})$

$\Rightarrow \mathcal{F}(S_1) \subseteq \mathcal{B}(\mathbb{R})$

also, any open set in $\mathbb{R}$ is of form $\bigcup_{i=1}^{\infty} (a_i, b_i)$

then as $(a_i, b_i) \in S_1 \ \forall i$

$\Rightarrow \bigcup_{i=1}^{\infty} (a_i, b_i) \in \mathcal{F}(S_1)$

$\Rightarrow \mathcal{F}(\{\bigcup_{i=1}^{\infty} (a_i, b_i)\}) \subseteq \mathcal{F}(S_1)$

$\Rightarrow \mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}(S_1)$

so, $\mathcal{B}(\mathbb{R}) = \mathcal{F}_1(S_1)$

(b) $S_2 = \{[a,b] \mid a<b\}$

then $[a,b] = \underbrace{\left((-\infty,a) \cup (a,\infty)\right)^c}_{\in \mathcal{F}_1(S_1)}$ ($(-\infty,a) \in S_1$, $(a,\infty) \in S_1$)

$\Rightarrow \mathcal{F}_1(S_2) \subseteq \mathcal{F}_1(S_1)$

also, $(a,b) = \bigcup_{n=1}^{\infty} \underbrace{[a-\frac{\varepsilon}{n}, b-\frac{\varepsilon}{n}]}_{\in \mathcal{F}_1(S_2)}$ (ε small so $\subset$ $\supset$)

$\Rightarrow \mathcal{F}_1(S_1) \subseteq \mathcal{F}_1(S_2)$

(c) $(a,b] = \left(\bigcup_{n=1}^{\infty} \underbrace{(-\infty, a-\frac{1}{n}) \cup (b,\infty)}_{\in \mathcal{F}_1(S_1)}\right)^c$

$(a,b) = \bigcup_{n=1}^{\infty} \underbrace{(a, b-\frac{\varepsilon}{n}]}_{\in \mathcal{F}_1(S_3)}$ (ε small)

(d) $[a,b)$ → same as above

(e) $(a,+\infty)$ same as above

(f) $(-\infty,b)$ same as above

(g) $[a,\infty)$ same as above

(h) $(-\infty,b]$ same as above

10. $a \in \mathbb{R}$ $\delta_a : \mathcal{B}(\mathbb{R}) \longrightarrow [0,\infty]$

$$\delta_a(E) = \begin{cases} 1 ; & a \in E \\ 0 ; & a \notin E \end{cases}$$

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \delta_a)$ is a measure space

then $\delta_a(E)$ is a measure

① $\delta_a(\phi) = 0$ as $a \notin \phi$

② for $\{E_i^o\}_{i=1}^{\infty} \subseteq \mathcal{B}(\mathbb{R})$ s.t

$E_i^o \cap E_j^o = \phi$ $\quad E = \bigcup_{i=1}^{\infty} E_i^o$

if $a \in E$ then $\exists$ unique $i \in \mathbb{N}$ s.t $a \in E_i^o$ as all disjoint

$\Rightarrow \delta_a(E) = 1 = \sum_{i=1}^{\infty} \delta_a(E_i^o) = 1$

if $a \notin E \Rightarrow a \notin E_i \ \forall i \Rightarrow \delta_a(E) = 0 = \sum_{i=1}^{\infty} \delta_a(E_i)$

δ_a is a σ-additive function on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$
($\mathbb{R}$: Ω; $\mathcal{B}(\mathbb{R})$: σ-algebra on $\mathbb{R}$)

$$\mathcal{B}(\mathbb{R})_{\delta_a} = \{A \cup N \mid A \in \mathcal{B}(\mathbb{R}),\ N \subseteq F \text{ s.t } \delta_a(F)=0\}$$
$$= \{A \cup N \mid A \in \mathcal{B}(\mathbb{R}),\ N \in \mathcal{P}(\mathbb{R}\setminus\{a\})\}$$
$$= \mathcal{P}(\mathbb{R}\setminus\{a\}) \cup \{\phi, a\}$$
$$= \mathcal{P}(\mathbb{R})$$

$a \in \mathbb{R}^d$ $\quad \delta_a : \mathcal{B}(\mathbb{R}^d) \to [0,\infty]$

$$\delta_a(E) = \begin{cases} 1; & a \in E \\ 0; & a \notin E \end{cases}$$

then ① $\delta_a(\phi) = 0$ (trivial)

② $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{B}(\mathbb{R}^d)$
s.t $E_i^\circ \cap E_j^\circ = \phi$

$$E = \bigcup_{i=1}^{\infty} E_i^\circ$$

if $a \in E$, then $\exists$ unique i s.t $a \in E_i^\circ$

$$\Rightarrow \delta_a(E) = \sum_{i=1}^{\infty} \delta_a(E_i)$$

if $a \notin E$ then trivially true

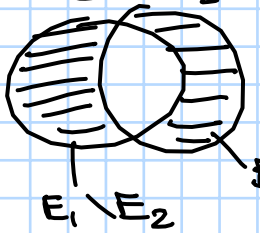
$\therefore (\mathbb{R}, \mathcal{B}(\mathbb{R}^d), \delta_a)$ is measure space

$$\mathcal{B}(\mathbb{R}^d)_{\delta_a} := \{A \cup N \mid A \in \mathcal{B}(\mathbb{R}^d),\ N \subseteq F,\ \delta_a(F)=0\}$$
$$= \{A \cup N \mid A \in \mathcal{B}(\mathbb{R}^d),\ N \in \mathcal{P}(\mathbb{R}^d \setminus \{a\})\}$$
$$= \mathcal{P}(\mathbb{R}^d \setminus \{a\}) \cup \{0, a\}$$
$$= \mathcal{P}(\mathbb{R}^d)$$

11. (X, M, μ) measure space

$$E_1 \Delta E_2 = [E_1 \setminus E_2] \cup [E_2 \setminus E_1]$$

(a) $E_1, E_2 \in M$



$(E_1 \setminus E_2) \cap (E_2 \setminus E_1) = \phi$
($E_1 \setminus E_2 \in M$, $E_2 \setminus E_1 \in M$)

$$\Rightarrow \mu(E_1 \setminus E_2) + \mu(E_2 \setminus E_1) = 0$$
$$\Rightarrow \mu(E_1 \setminus E_2) = 0,\ \mu(E_2 \setminus E_1) = 0$$

now, $\mu(E_1 \cup E_2) = \mu((E_1 \Delta E_2) \cup (E_1 \cap E_2))$

$E_1 \Delta E_2 \in M$, $E_1 \cap E_2 \in M$

disjoint

$\mu(E_1 \cup E_2) = 0 + \mu(E_1 \cap E_2)$

but $\mu(E_1 \cap E_2) \leq \mu(E_1) \leq \mu(E_1 \cup E_2)$

$\mu(E_2)$

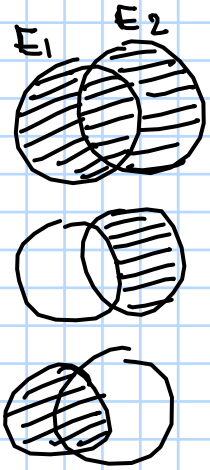
$\Rightarrow \mu(E_1 \cap E_2) = \mu(E_1) = \mu(E_2) = \mu(E_1 \cup E_2)$

$\Rightarrow \mu(E_1) = \mu(E_2)$

(b) μ is complete wrt M, i.e

$\forall A \subseteq E \in M$ s.t $\mu(E) = 0$

$\Rightarrow A \in M$



also $\mu : M \to [0, \infty]$

$\mu(E_1 \Delta E_2) = 0$

$\Rightarrow E_1 \Delta E_2$ in domain of M

$E_1 \Delta E_2 \in M$

then

$E_1 \setminus E_2 \subseteq E_1 \Delta E_2$

$\mu(E_1 \Delta E_2) = 0$

$\Rightarrow E_1 \setminus E_2 \in M$

$\Rightarrow E_1 \cap E_2 \in M$

$\Rightarrow (E_1 \cap E_2) \cup (E_2 \setminus E_1) \in M$

$\Rightarrow E_2 \in M$

12. $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$

$\mathcal{A}(S)$ $S = \{(a, b] \mid -\infty \leq a < b < \infty\} \cup \{\phi\}$

$\lambda : S \to [0, \infty]$

↳ length function (σ-finite)

$\nu : \mathcal{A}(S) \to [0, \infty]$

↳ unique extension (σ-finite)

$\pi^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$

↑

outer measure is Lebesgue measure

$\mathcal{L} = \{ A \subseteq \Omega \mid \pi^*(E) = \pi^*(E \cap A) + \pi^*(E \cap A^c) \}$

$\forall E \subseteq \Omega$

(a) $A \in \mathcal{L}$

$$\pi^*(A) = \inf\left\{ \sum_{j=1}^{\infty} \nu(I_j) \;\middle|\; A \subseteq \bigcup_{j=1}^{\infty} I_j,\ I_j \subseteq \mathbb{R} \text{ is open interval for all } j \in \mathbb{N} \right\}$$

$$\pi^*(A) = \inf\left\{ \sum_{j=1}^{\infty} \nu(E_j) \;\middle|\; A \subseteq \bigcup_{j=1}^{\infty} E_j,\ E_j \in \mathcal{A}(S) \ \forall j \right\}$$

now, $A \in \mathcal{A}(S)$ iff $A = \bigcup_{i=1}^{n} E_i$ & $\exists \{E_i\}_{i=1}^{n} \subseteq S$ $E_i \cap E_j = \phi$

$E_j \in \mathcal{A}(S) \Rightarrow \exists \{E_{i,j}\}_{i=1}^{n_j}$ s.t $\{E_{i,j}\}_{i=1}^{n_j} \subseteq S$

$E_{i,j} \cap E_{k,j} = \phi$

let $E_{i,j} = (a_{i,j}, b_{i,j}]$

then, $A \subseteq \bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} \left(\bigcup_{i=1}^{n_j} (a_{i,j}, b_{i,j}] \right)$

$$\subseteq \bigcup_{j=1}^{\infty} \left(\bigcup_{i=1}^{n_j} (a_{i,j}, b_{i,j} + \tfrac{\varepsilon}{2^j}) \right)$$

for some $\varepsilon > 0$ ↳ as finite

wlog only one I_j as all finite let $I_j = \bigcup_{i=1}^{n_j} (a_{i,j}, b_{i,j} + \tfrac{\varepsilon}{2^j})$

then, $\sum_{j=1}^{\infty} \nu(E_j) \leq \sum_{j=1}^{\infty} \nu(I_j) \leq \sum_{j=1}^{\infty} \nu(E_j) + c\varepsilon$

for some $c > 0$

$$\Rightarrow \inf\left\{ \sum_{j=1}^{\infty} \nu(E_j) \right\} \leq \inf\left\{ \sum_{j=1}^{\infty} \nu(I_j) \right\} \leq \inf\left\{ \sum_{j=1}^{\infty} \nu(E_j) \right\} + c\varepsilon$$

$\varepsilon \to 0$

$$\Rightarrow \pi^*(A) = \inf\left\{ \sum \nu(E_j) \right\} = \left\{ \sum \nu(I_j) \right\}$$

(b) $x \in \mathbb{R}$ $\lambda(\{x\}) = 0$

now, $\lambda = \pi^*$

$$\pi^*(\{x\}) = \inf\left\{ \sum_{j=1}^{\infty} \nu(E_j) \;\middle|\; \{x\} \subseteq \bigcup_{j=1}^{\infty} E_j,\ E_j \in \mathcal{A}(S) \right\}$$

$$\leq \sum_{j=1}^{\infty} \nu(E_{\varepsilon,j}) \quad \text{for } \varepsilon > 0$$

$E_{\varepsilon,j} = [x - \tfrac{\varepsilon}{2^j}, x + \tfrac{\varepsilon}{2^j})$ $\forall j$

$\in S$ and $\mathcal{A}(S)$

and $\bigcup_{i=1}^{j} E_{\varepsilon,i} = [x - \varepsilon, x + \varepsilon) \supseteq \{x\}$

$$\nu(E_{\varepsilon,i}) = \lambda(E_{\varepsilon,i}) = (x + \tfrac{\varepsilon}{2^i}) - (x - \tfrac{\varepsilon}{2^i}) = \frac{\varepsilon}{2^{i-1}}$$

$$\sum_{i=1}^{\infty} \nu(E_{\varepsilon,i}) = 2\varepsilon$$

$$\Rightarrow \pi^*(\{x\}) \leq 2\varepsilon \Rightarrow \pi^*(\{x\}) = 0 \text{ as } \varepsilon \to 0$$

(c) $A \subset \mathbb{R}$ ↑ countable $\quad \pi^*(A) = 0$

now, as $A = \{x_1, x_2, \dots\}$ ← countable

$$\pi^*(A) = \pi^*(\{x_1, x_2, \dots\}) \leq \sum_{i=1}^{\infty} \pi^*(\{x_i\}) \quad (\because \pi^* \text{ is outer measure})$$

$\Rightarrow \pi^*(A) \leq 0$ ($\because$ above)

$\Rightarrow \pi^*(A) = 0$

(d) $t + A := \{t + a \mid a \in A\}$

as $A \in \mathcal{L}$, $\exists B \in \mathcal{B}(\mathbb{R})$ s.t

$A \subseteq B$ and $\pi^*(A) = \pi^*(B)$

now, $t + A \subseteq t + B$ also $\pi^*(B \setminus A) = 0$



as for any $B \in \mathcal{B}(\mathbb{R})$ countable union \ intersection \ complement of open sets

$\Rightarrow t + B \in \mathcal{B}(\mathbb{R})$

so, $t + A \subseteq t + B \rightarrow \pi^*(t + B) \leq \sum(I_i) \leq \pi^*(t + B) + \varepsilon$

$\pi^*(B) \leq \sum(I_i) \leq \pi^*(B) + \varepsilon$

$t + B \in \mathcal{B}(\mathbb{R})$

also $(t + B) \setminus (t + A) \subseteq t + (B \setminus A)$ → $\pi^*(B) = \pi^*(t + B)$

as $\forall x \in t + B$, $x \notin t + A$

$x - t \in B$, $x - t \notin A$

$x - t \in B \setminus A$

$\Rightarrow x \in t + B \setminus A$

now $\pi^*(B \setminus A) = 0$

and $\forall \varepsilon > 0$, $\exists \{I_i^o\}$ s.t ← open

$$\pi^*(B \setminus A) \leq \sum_{i=1}^{\infty} \nu(I_i) \leq \pi^*(B \setminus A) + \varepsilon$$

$$\Rightarrow \pi^*(t + B \setminus A) \leq \sum_{i=1}^{\infty} \nu(t + I_i^o) = \sum_{i=1}^{\infty} \nu(I_i^o) \leq \pi^*(B \setminus A) + \varepsilon$$

$\pi^*(B \setminus A) = 0$

$\varepsilon \to 0$

$\Rightarrow \pi^*(t + B \setminus A) = 0$

now, $\Rightarrow \pi^*((t+B)\setminus(t+A)) = 0$

$\subseteq (t+B\setminus A)$

now, $t+A \subseteq t+B$

$\pi^*(t+B) = \pi^*(B)$



$\Rightarrow (t+A)\cup(t+B\setminus t+A) = t+B$

$\Rightarrow \pi^*(t+A) = \pi^*(t+B) = \pi^*(B)$

(e) $A \in \mathcal{L} \Rightarrow \exists B \in \mathcal{B}(\mathbb{R})$ s.t $\pi^*(A) = \pi^*(B)$, $A \subseteq B$

now, $t B \in \mathcal{B}(\mathbb{R})$

$t A \subseteq t B$

$\pi^*(B\setminus A) = 0$

$(t B)\setminus(t A) \subseteq t(B\setminus A)$

$\Rightarrow \pi^*(t(B\setminus A)) = 0$

$\Rightarrow \pi^*(tB\setminus tA) = 0$

$\Rightarrow \pi^*(tB) = \pi^*(tA)$

$|t|\,\pi^*(B)$

$|t|\,\pi^*(A)$

13. $(\Omega, \mathcal{H}, P)$

$P(\Omega) = 1$

$\Omega \subseteq \mathbb{N}$, $\mathcal{H} = 2^{\Omega} = \mathcal{P}(\Omega)$

(a) $\Omega = \{H, T\}$ $P(\phi) = 0$

$P(\{H\}) = \frac{1}{2}$

$P(\{T\}) = \frac{1}{2}$

$P(\Omega) = 1$

then $\mathcal{H} = 2^{\Omega}$ finite set also σ-algebra on 2^{Ω}

now, $(\Omega, \mathcal{H}, P)$

then ① $P(\phi) = 0$

② for any $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ $E_i \cap E_j = \phi$

then $\mathbb{P}(\bigcup_{i=1}^{\infty} E_i)$

always finite, as Ω finite

$\bigcup_{i=1}^{\infty} E_i = \bigcup_{j=1}^{n} E_{i_j}$ ← finite many

and as disjoint if $n=0$ then trivial

$\mathbb{P}(\bigcup E_i) = 0 = \sum \mathbb{P}(E_i)$

$n=1$:
also trivial → 3 cases

$n=2$:
$E_1 = H$, $E_2 = T$
also trivial

so, $\mathbb{P}$ is a σ-additive function

& $\mathbb{P}(\Omega) = 1 \Rightarrow \mathbb{P}$ is probability measure

(b) $N \in \mathbb{N}$ $p \in (0,1)$

$\Omega = \{0, 1, \dots N\}$

$\mathbb{P}(\{k\}) = \binom{N}{k} p^k (1-p)^{N-k}$

$\bigcup E_i = \{k_1, k_2, \dots k_n\}$

then trivial as everything finite

$$\mathbb{P}(\{1,2,\dots N\}) = \sum_{k=1}^{N} \binom{N}{k} p^k (1-p)^{N-k}$$

$$= (1-p+p)^N = 1^N = 1$$

(c) $\lambda > 0$, $\Omega = \mathbb{N}$

$$\mathbb{P}(\{k\}) = \frac{e^{-\lambda} \lambda^k}{k!}$$

same trivially as everything countable

$$\mathbb{P}(\mathbb{N}) = \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!}$$

$$= e^{-\lambda} \left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \right] = e^{-\lambda} e^{\lambda} = 1$$

even, $\mathbb{P}(A)$ ↑ odd integers

$$\mathbb{P}(A) = \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^{2n}}{(2n)!}$$

$$\sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} = \cosh(\lambda) = \frac{e^{\lambda}+e^{-\lambda}}{2}$$

$$\sum_{k=0}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!} = \sinh(\lambda) = \frac{e^{\lambda}-e^{-\lambda}}{2}$$

$$e^{-\lambda}\left(\frac{e^{\lambda}-e^{-\lambda}}{2}\right) = \frac{1-e^{-2\lambda}}{2}$$

no closed form for primes

1. $\mathcal{F}_\mu := \{A \cup N \mid A \in \mathcal{F}, N \subseteq E,\ E \in \mathcal{F},\ \mu(E)=0\} \subseteq \mathcal{P}(\Omega)$

(a) $\mathcal{F} \subseteq \mathcal{F}_\mu$ is trivial when $N = \emptyset$

then ① $\emptyset, \Omega \in \mathcal{F}_\mu$

② $\tilde{A} \in \mathcal{F}_\mu$

$\tilde{A} = A \cup N$, $N \subseteq F$, $\mu(F) = 0$

now, $(\tilde{A})^c = A^c \cap N^c$



$= A^c \cap (F \setminus N \cup F^c)$

$= (A^c \cap F^c) \cup (A^c \cap (F \setminus N))$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A^c \cap (F \setminus N) \subseteq F$

$A^c \cap F^c \in \mathcal{F} \Rightarrow (\tilde{A})^c \in \mathcal{F}_\mu$

③ $\{\tilde{A}_i\}_{i=1}^{\infty} \in \mathcal{F}_\mu$

$\{A_i\}_{i=1}^{\infty} \in \mathcal{F}$ $\quad \{N_i\} \subseteq \{F_i\}$

$\bigcup \tilde{A}_i = (\underbrace{\bigcup A_i}_{\in \mathcal{F}}) \cup (\underbrace{\bigcup N_i}_{\subseteq \bigcup F_i})$, $\mu(\bigcup F_i) = 0$

$\Rightarrow \bigcup \tilde{A}_i \in \mathcal{F}_\mu$

(b) $\bar{\mu} : \mathcal{F}_\mu \to [0,\infty]$

$\bar{\mu}(A \cup N) = \mu(A)$

$\bar{\mu}|_{\mathcal{F}} = \bar{\mu}(A \cup \phi)$

$= \mu(A) \quad \forall A \in \mathcal{F}$

$= \mu$

now, ① $\bar{\mu}(\phi) = 0$

② for $\{A_i\}_{i=1}^{\infty}$ disjoint

$\exists \{A_i\}_{i=1}^{\infty}, \{N_i\}_{i=1}^{\infty}$ disjoint

s.t

$\bar{\mu}(\bigcup_{i=1}^{\infty} A_i) = \bar{\mu}((\bigcup A_i) \cup (\bigcup N_i))$

$= \mu(\bigcup A_i)$

$= \sum \mu(A_i)$

$= \sum \mu(A_i \cup N_i)$

$= \sum \bar{\mu}(A_i)$

(c) $(\Omega, \mathcal{F}_\mu, \bar{\mu})$ completion

① it is complete

② $\mathcal{F}_\mu \supseteq \mathcal{F}$

$\bar{\mu}|_{\mathcal{F}} = \mu$

③ for any other complete space

$(\Omega, \mathcal{F}_1, \mu_1)$

s.t $\mathcal{F} \subseteq \mathcal{F}_1, \quad \mu_1|_{\mathcal{F}} = \mu$

then $\forall F \in \mathcal{F}$ s.t

$\mu(F) = 0$

$\Rightarrow F \in \mathcal{F}_1$

$\bar{\mu}(F) = 0$

now, $\forall E \subseteq F \Rightarrow E \in \mathcal{F}_\mu$

and so, $\forall A \in \mathcal{F}_1, E \subseteq F \in \mathcal{F}$

$\Rightarrow A \cup E \in \mathcal{F}_1$

$\Rightarrow \mathcal{F}_\mu \subseteq \mathcal{F}_1$

also, $\mu_1|_{\mathcal{H}_\mu} = \mu_1(A \cup E)$

$= \mu_1(A) + \mu_1(E \setminus A)$ $\quad A \subseteq E \subseteq F$

$= \mu_1(A) \quad 0$

$= \mu(A)$

$= \bar{\mu}(A \cup E)$

$\mu_1|_{\mathcal{H}_\mu} = \bar{\mu}$

2. $\{f_n\}_{n=1}^{\infty}$ $f_n : \Omega \to [0,\infty]$ $\forall n \in \mathbb{N}$, simple

$g : \Omega \to [0,\infty]$ simple

$f_n(\omega) \leq f_{n+1}(\omega) \; \forall \omega \in \Omega \; \forall n \in \mathbb{N}$

$g(\omega) \leq \lim_{n\to\infty} f_n(\omega) \; \forall \omega \in \Omega$

now, $g(\omega) \leq \lim_{n\to\infty} f_n(\omega) \quad \forall \omega \in \Omega$

$\downarrow$

$\exists \{E_i^\circ\}_{i=1}^{n}$ disjoint $\{c_i\} \subseteq \mathbb{R}_{\geq 0}$

s.t. $\Omega = \bigcup_{i=1}^{n} E_i$

$g = \sum_{i=1}^{n} c_i^\circ \chi_{E_i^\circ}$

let $\{\underbrace{E_i^\circ \cap F_{k,j}^\circ}_{\text{disjoint}}\}_{k=1,\,i=1}^{n_j,\,n}$

$\{c_{k,j}^\circ \text{ for } i\}$

$\forall \omega \in E_i^\circ$

$c_i^\circ \leq \lim_{j\to\infty} \sum_{k=1}^{n_j} \underbrace{c_{k,j}^\circ \chi_{F_{k,j}^\circ}(\omega)}_{\text{disjoint}}$

let $\tilde{c}_j^\circ = \min_k (c_{k,j}^\circ)$

$\Rightarrow c_i^\circ \leq \lim_{j\to\infty} \tilde{c}_j^\circ$

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t.

$c_i^\circ \leq \tilde{c}_N + \varepsilon$

$\Rightarrow \sum c_i^\circ \mu(E_i^\circ \cap F_{N,j}^\circ) \leq \sum \tilde{c}_N \mu(E_i^\circ \cap F_{N,j}^\circ) + \varepsilon \mu(E_i^\circ \cap F_{N,j})$

$\leq \sum c_{N,j}^\circ \mu(E_j^\circ \cap F_{N,j}^\circ) + \varepsilon \mu(E_i^\circ)$

$\Rightarrow c_i^\circ \mu(E_i^\circ) \leq \sum c_{N,j}^\circ \mu(E_j^\circ \cap N_{N,j}^{i,j}) + (\;)$

$\Rightarrow \int_\Omega g\,d\mu \leq \int_\Omega f_N\,d\mu + \varepsilon M(\Omega)$

if all $\mu(E_i) = \infty$

$\Rightarrow \int_\Omega g\,d\mu = \infty$

else if $\mu(E_i \cap F_{i,\beta}) = \infty$

↓

trivially true

3. $f : \Omega \to [0,\infty]$

↳ measurable function

$$I(f) := \sup\left\{ \int_\Omega g\,d\mu \;\middle|\; g \text{ is basic}, \ 0 \leq g \leq f \right\}$$

$\exists \{f_n\}$ basic

$f_n \leq f_{n+1}$ $f_n \to f$ $n \to \infty$

g is basic

$I(f) \geq \int_\Omega f_n\,d\mu$

as $f_n \leq f$

↑

Basic

$\Rightarrow I(f) \geq \lim_{n\to\infty} \int_\Omega f_n\,d\mu = \int_\Omega f\,d\mu$

$\forall \varepsilon > 0$, $\exists$ basic g s.t. $g \leq f$

$I(f) \geq \int g \geq I(f) - \varepsilon$

$\int_\Omega g\,d\mu \geq I(f) - \varepsilon$

now $g \leq f \Rightarrow \int_\Omega g\,d\mu \leq \int_\Omega f\,d\mu$

$\Rightarrow I(f) - \varepsilon \leq \int_\Omega f\,d\mu$

$\int_\Omega f\,d\mu \geq I(f)$

$\Rightarrow \varepsilon \to 0$ $\int_\Omega f\,d\mu \geq I(f)$

$\int_\Omega f\,d\mu \leq I(f)$

$\Rightarrow I(f) = \int_\Omega f\,d\mu$

Tutorial-5:

1. $f, g: \Omega \to [0,\infty)$ → basic

$\alpha \in [0,\infty)$

$f = \sum_{i=1}^{n} c_i \chi_{E_i}$

$g = \sum_{j=1}^{m} d_j \chi_{F_j}$

$\{E_i \cap F_j\}$ partition of Ω

then, $x \in \Omega$, $\exists\, i_x, j_x$ s.t

$x \in E_{i_x} \cap E_{j_x}$

$f(x) = c_{i_x}$ $\quad m_{c_i, d_j} > 0$

$E_{i_x} = E_{i_x} \cap (\bigcup_{j=1} F_j)$

$= \bigcup_{j=1}^{m} (E_{i_x} \cup F_j)$

$\exists\, j_x$

$f(x) = c_{i_x} \chi_{E_{i_x} \cap F_{j_x}}$

$f = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i \chi_{E_i \cap F_j}$

$g = \sum_{j=1}^{m} \sum_{i=1}^{n} d_j \chi_{E_i \cap F_j}$

$f + g = \sum_{j=1}^{m} \sum_{i=1}^{n} (c_i + d_j) \chi_{E_i \cap F_j}$

$\Rightarrow \int_{\Omega} f+g \, d\mu = \sum_{j=1}^{m} \sum_{i=1}^{n} (c_i + d_j)\, \mu(E_i \cap F_j)$

$= \sum\sum c_i \mu(E_i \cap F_j) + \sum\sum d_j \mu(E_i \cap F_j)$

$= \int_{\Omega} f \, d\mu + \int_{\Omega} g \, d\mu$

same for αg

(b) $f \le g$

$\Rightarrow g - f \ge 0$

$g = g - f + f$

$\int_{\Omega} g \, d\mu = \int_{\Omega} \underbrace{(g-f)}_{g-f \ge 0} d\mu + \int_{\Omega} f \, d\mu \ge 0 + \int_{\Omega} f \, d\mu$

2. $(\Omega, \mathcal{H})$ $f, g : \Omega \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$

$\alpha \in \mathbb{R}$

$f^+(x) = \max\{0, f(x)\}$

$f^-(x) = -\min\{0, f(x)\}$

$f^+ - f^- = f$

$|f| = f^+ + f^-$

$f^{-1}(B) \in \mathcal{H} \ \forall \ B \in \overline{\mathcal{B}}$

$\Leftrightarrow f^{-1}([-\infty, x)) \in \mathcal{H} \ \forall \ x \in \mathbb{R}$

$\Leftrightarrow f^{-1}([-\infty, x]) \in \mathcal{H} \ \forall \ x \in \mathbb{R}$

$\Leftrightarrow f^{-1}((x, \infty]) \in \mathcal{H} \ \forall \ x \in \mathbb{R}$

$\Leftrightarrow f^{-1}([x, \infty]) \in \mathcal{H} \ \forall \ x \in \mathbb{R}$

$$(\alpha + f)^{-1}([-\infty, x]) = \{\omega \in \Omega \mid (\alpha + f)(\omega) \leq x\}$$
$$= \{\omega \in \Omega \mid f(\omega) \leq x - \alpha\}$$
$$= f^{-1}([-\infty, x - \alpha])$$
$$\in \mathcal{H} \text{ as } f \text{ is measurable}$$

αf

$\alpha = 0$ trivial

wlog $\alpha > 0$

$$\alpha f^{-1}([-\infty, x]) = \{\omega \in \Omega \mid \alpha f(\omega) \leq x\}$$
$$= \{\omega \in \Omega \mid f(\omega) \leq \frac{x}{\alpha}\}$$
$$= f^{-1}([-\infty, \tfrac{x}{\alpha}])$$
$$\in \mathcal{H}$$

$$(f+g)^{-1}([-\infty, x)) = \{\omega \in \Omega \mid f(\omega) + g(\omega) < x\}$$
$$= \bigcup_{r \in \mathbb{Q}} f^{-1}([-\infty, r)) \cap g^{-1}([-\infty, x - r))$$
$$\in \mathcal{H} \qquad \in \mathcal{H} \qquad \in \mathcal{H}$$

$(f^2)^{-1}([-\infty, x])$

$= \{\omega \in \Omega \mid f^2(\omega) \leq x\}$

$x > 0 \to f^{-1}([-x, x]) \in \mathcal{H}$

$x \leq 0 \to f^{-1}(\{0\}) \in \mathcal{H}$ as $\{0\} \in \mathcal{B}(\overline{\mathbb{R}})$

$$fg = \frac{1}{2}\underbrace{[(f+g)^2 - f^2 - g^2]}_{\text{all measurable}}$$

$f^+(x) = \max(0, f(x))$

$$(f^+)^{-1}([-\infty, x)) = \begin{cases} \emptyset & ; x \le 0 \\ f^{-1}([-\infty, x)) & ; x > 0 \end{cases}$$

same for rest

3. $f_n : \Omega \to \mathbb{R} \cup \{\pm\infty\}$

(a) $(\sup f_n)^{-1}([-\infty, x)) = \{\omega \in \Omega \mid \sup(f_n(x)) < x\}$

$= \{\omega \in \Omega \mid f_n(x) < x \ \forall\, n \in \mathbb{N}\}$

$= \bigcap_{n=1}^{\infty} f_n^{-1}([-\infty, x))$

$\in \mathcal{F}$

same for inf

(b) $\overline{\lim_{n\to\infty}} f_n \qquad \underline{\lim_{n\to\infty}} f_n$

now, $\left(\overline{\lim_{n\to\infty}} f_n\right)^{-1}([x, \infty]) = \left\{\omega \in \Omega \mid \overline{\lim_{n\to\infty}} f_n(\omega) \ge x\right\}$

$$\overline{\lim_{n\to\infty}} f_n = \lim_{n\to\infty} \sup_{k \ge n} f_k$$

$$= \inf_{n \ge 1} \sup_{k \ge n} f_k \ge x$$

$$\Rightarrow \sup_{k \ge n} f_k \ge x \ \forall\, n \in \mathbb{N}$$

$$\Rightarrow \underbrace{\sup_n f_n \ge x}_{\downarrow}$$

$$\bigcup_{n=1}^{\infty} f_n^{-1}([x, \infty])$$

$$\Rightarrow \left(\overline{\lim_{n\to\infty}} f_n\right)^{-1} = \bigcup_{n=1}^{\infty} f_n^{-1}([x, \infty]) \in \mathcal{F}$$

$\in \mathcal{F}$

$$\underline{\lim_{n\to\infty}} f_n = \lim_{n\to\infty} \inf_{k \ge n} f_k$$

$$= \sup_{n \in \mathbb{N}} \inf_{k \ge n} f_k$$

(c) $\{f_n\} \longrightarrow f$

$$\overline{\lim_{n\to\infty}} f_n = \underline{\lim_{n\to\infty}} f_n = \lim_{n\to\infty} f_n = f$$

measurable

$\Rightarrow f$ is measurable

Tutorial-6:

1. $\{a_n\}_{n=1}^{\infty}$, $a_n \geq 0$ $\quad f : \mathbb{N} \to [0, \infty)$

$f(n) = a_n \quad \forall n \in \mathbb{N}$

$$f_K(n) = \begin{cases} f(n) ; & n \leq K \\ 0 ; & n > K \end{cases}$$

then $f_K = \sum_{i=1}^{K} a_i \chi_{\{i\}} + 0 \chi_{\{K+1, K+2, \ldots\}}$

$f_K \leq f_{K+1}$

$\lim_{K \to \infty} f_K(n) = f(n)$

$$\Rightarrow \int_{\mathbb{N}} f \, d\mu = \lim_{n \to \infty} \int_{\mathbb{N}} f_n \, d\mu$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} a_i$$

$$= \sum_{i=1}^{\infty} a_i$$

2. $\Omega \neq \phi$ $\quad (\Omega, \mathcal{F}, \mu) \to$ measure space

$f_n : \Omega \to [0, \infty]$

$\uparrow$ measurable $\forall n \in \mathbb{N}$

$f_n \geq f_{n+1} \quad \forall n \in \mathbb{N} \quad f_n(\omega) \to f(\omega)$ as $n \to \infty$

& if f_1 is Lebesgue integrable

$$\int_{\Omega} f_1 \, d\mu < \infty$$

then $\lim_{n \to \infty} \int_{\Omega} f_n \, d\mu = \int_{\Omega} f \, d\mu$

$g_n := f_1 - f_n : \Omega \to [0, \infty]$

as $f_1 \geq f_n$

now, ① $f_1 - f_n$ is measurable

② $g_n \geq 0$

$f_n \geq f_{n+1}$

$\Rightarrow -f_n \leq -f_{n+1}$

$\Rightarrow f_1 - f_n \leq f_1 - f_{n+1}$

$\Rightarrow g_n \leq g_{n+1}$

$\lim_{n\to\infty} g_n = \lim_{n\to\infty} (f_1 - f_n)$

$= f_1 - f = g$ say

by MCT: $\lim_{n\to\infty} \int_\Omega g_n \, d\mu = \int_\Omega g \, d\mu$

$\Rightarrow \int_\Omega f_1 \, d\mu - \lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f_1 \, d\mu - \int_\Omega f \, d\mu$

as $\int_\Omega f_1 \, d\mu < \infty$

$\Rightarrow \lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu$

now if $((0,1), \mathcal{B}(0,1), \lambda)$

$f_n : (0,1) \longrightarrow [0,\infty]$

$f_n(x) = \frac{1}{x} \chi_{(0,\frac{1}{n})}$

$f_1(x) = \chi_{(0,1)}$

$\int_{(0,1)} f_1(x) dx = \int_0^1 \frac{1}{x} dx = \infty$

not integrable

also, $f_n(x) = \frac{1}{x} \chi_{(0,\frac{1}{n})}$

so, $f_{n+1}(x) \leq f_n(x) \quad \forall x \in (0,1)$

also

$\lim_{n\to\infty} f_n \equiv 0$

$\int_{(0,1)} f_n \, d\mu = \int_0^{y_n} \frac{1}{x} dx$

$= \mu\left(\frac{1}{n}\right) - \mu(0^+)$

$= \infty$

$\Rightarrow \lim_{n\to\infty} \int_{(0,1)} f_n \, d\lambda = \infty$

$\int_{(0,1)} f \, d\lambda = 0$

3. strict inequality holds in Fatou's lemma

$f_n : \Omega \to [0, \infty]$ measurable

$$\int_\Omega \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n \, d\mu$$

$f_n = \chi_{[2,3]}$ n is even

$f_n = \chi_{[0,1]}$ n is odd

$\int_{\mathbb{R}} f_n \, d\lambda = 1$ for n odd / even

$\Rightarrow \liminf_{n\to\infty} \int_{\mathbb{R}} f_n = 1$

$\liminf_{n\to\infty} f_n = 0$

$\Rightarrow \int_{\mathbb{R}} 0 \, d\lambda = 0 < 1$ ← strict inequality

1. $f : \mathbb{R} \to \mathbb{R}$ monotone with $x < y$, $f(x) \leq f(y)$

$f^{-1}((-\infty, x]) = \{\omega \in \mathbb{R} \mid f(\omega) \leq x\}$

$= \{\omega \leq f^{-1}(x)\}$

$= (-\infty, f^{-1}(x)]$ ← cases

$\in \mathcal{B}(\mathbb{R})$

$\Rightarrow f^{-1}((-\infty, x]) \in \mathcal{B}(\mathbb{R})$

$\Rightarrow f$ is measurable

2. $f^{-1}(\alpha, \infty)$

$\alpha \in \mathbb{R}$ $r_1 < r_2 < \dots$

s.t. $\lim_{n\to\infty} r_n = \alpha$

then $\bigcup_{i=1}^{\infty} (r_i, \infty) = (\alpha, \infty)$

$f^{-1}(\alpha, \infty) = f^{-1}\left(\bigcup_{i=1} (r_i, \infty)\right)$

3. $\{x \in \mathbb{R} \mid f(x) = \alpha\} = f^{-1}(\{\alpha\})$

$= f^{-1}([-\infty, \alpha]) \setminus f^{-1}([-\infty, \alpha))$

$\in \mathcal{B}(\mathbb{R}) \qquad \in \mathcal{B}(\mathbb{R})$

$\in \mathcal{B}(\mathbb{R})$

4. $f : [0,1] \to \mathbb{R}$

$\{x \in [0,1] \mid f(x) = c\} \in \mathcal{B}(\mathbb{R}) \quad \forall c \in \mathbb{R}$

true $\quad f^{-1}(\{c\}) \in \mathcal{B}(\mathbb{R}) \quad \forall c \in \mathbb{R}$

$f(x) = \chi_A(x) + x$

$A \notin \mathcal{B}(\mathbb{R})$

true $\quad \{x \in [0,1] \mid \underbrace{\chi_A(x) + x = c}\}$

c or c or $\emptyset$

so measurable

but $\{x \in [0,1] \mid \chi_A(x) + x > x\}$

$= \{x \in [0,1] \mid \chi_A(x) > 0\}$

$= A \notin \mathcal{B}(\mathbb{R})$

5. $\emptyset \subset E \subset \mathbb{R}$

$E \in \mathcal{L}$

$\lambda(E) < \infty$

$f_E : \mathbb{R} \to \mathbb{R}$

$f(x) = \lambda(E \cap (-\infty, x]) \quad \forall x \in \mathbb{R}$

for $y \geq x$

$E \cap (-\infty, x] \subseteq E \cap (-\infty, y]$

$\Rightarrow \lambda(E \cap (-\infty, x]) \leq \lambda(E \cap (-\infty, y])$

$\Rightarrow f(x) \leq f(y)$

also $f(y) - f(x) = \lambda(E \cap (-\infty, y]) - \lambda(E \cap (-\infty, x])$

$\lambda((-\infty, x]) + \lambda((x, y]) = \lambda((-\infty, y])$

$\lambda(E \cap (-\infty, x]) + \lambda(E \cap (x, y]) = \lambda(E \cap (-\infty, y])$

$= \lambda(E \cap (x, y])$

$\leq (x, y]$

$\leq y - x$

6. $f : \mathbb{R} \to \mathbb{R}$

$f = \sum_{i=1}^{n} c_i \chi_{E_i}$

wlog $c_1 < c_2 < c_3 \dots < c_n$

then $f^{-1}(\{c_1\}) = \{x \in \mathbb{R} \mid f(x) = c_1\}$

$= E_1 \in \mathcal{B}(\mathbb{R})$

then we repeate

7. $\Omega \neq \emptyset$

$(\Omega, \mathcal{F})$ $f_n : \Omega \to \mathbb{R} \cup \{\pm\infty\}$

$\overline{\lim_{n\to\infty}} f_n(x) = \underline{\lim_{n\to\infty}} f_n(x)$

$E = \{\omega \in \Omega \mid \overline{\lim} f_n(\omega) = \underline{\lim} f_n(\omega)\}$

$\overline{\lim} f_n$ measurable

$\underline{\lim} f_n$ measurable

$E = \{\omega \in \Omega \mid \overline{\lim} f_n(\omega) - \underline{\lim} f_n(\omega) = 0\}$

$f - g$ is measurable

$\Rightarrow (\overline{\lim} f_n - \underline{\lim} f_n)^{-1}(\{0\}) \in \mathcal{B}(\mathbb{R})$

8. $f : (0,1) \to \mathbb{R}$

↳ diff

$$f'(x) = \lim_{n\to\infty} \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$$

let $f_n(x) = \dfrac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$

$f_n \to f$ as $n \to \infty$

↑

measurable

↖ measurable

and $\lim_{n\to\infty} f_n = f$

9. let $f_n(x) = \left(1 + \frac{x}{n}\right)^n \chi_{[0,n]}$

$a_1 = a_2 = \dots = a_n = 1 + \frac{x}{n}$

$a_{n+1} = 1$

$$\left(1 + \frac{x}{n}\right)^{\frac{n}{n+1}} \leq \left[\frac{n\left(1 + \frac{x}{n}\right) + 1}{n+1}\right] = \left[\frac{n+1+x}{n+1}\right]$$

$$\Rightarrow \left(1 + \frac{x}{n}\right)^n \leq \left(1 + \frac{x}{n+1}\right)^{n+1}$$

$\forall x \in [0,n]$

$\Rightarrow f_n(x) \leq f_{n+1}(x)$

also $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

$f(x) = e^x$

$$\lim_{n\to\infty} \int_{[0,n]} f_n(x)\,dx = \int_0^\infty e^x\,dx$$

$$= \infty$$

10. $f: \Omega \to [0,\infty]$

↳ measurable fuction

$\int_\Omega f\,d\mu > 0$ iff $\mu(\{\omega \in \Omega \mid f(\omega) > 0\}) > 0$

($\Leftarrow$) $\mu(\{\omega \in \Omega \mid f(\omega) > 0\}) > 0$

$A_n = \{a \in \Omega \mid f(a) \geq \frac{1}{n}\}$

then $\{\omega \in \Omega \mid f(\omega) > 0\} = \bigcup_{i=1}^{\infty} A_i = A$

as $\mu(A) > 0$ $\quad \mu A_n \to \mu A$

$\exists n$ s.t $\mu(A_n) > 0$

$\Rightarrow f \geq \frac{1}{n} A_n$

$\Rightarrow \int f\,d\mu \geq \frac{1}{n} A_n = \frac{1}{n}\mu(A_n)$

> 0 as $\mu(A_n) > 0$

$\Rightarrow \int f\,d\mu > 0$

($\Rightarrow$) if $\mu(\{\omega \in \Omega \mid f(\omega) > 0\}) = 0$

then as f is measurable let

$g_n \uparrow f$

$\lim_{n\to\infty} g_n = f$

$g_n = \sum c_n \chi_{B_n} \leq f$

$\mu(\chi_{B_n}) = 0$ as $\chi_{B_n} \subseteq \{\omega \in \Omega \mid f(\omega) > 0\}$

$\Rightarrow \int g_n d\mu = 0$

$\forall n$

$\Rightarrow \int f d\mu = 0$ this is a contradiction

Tutorial-1:

2. $(\Omega, \mathcal{H}, \mu)$ measure space, $A, B \in \mathcal{H}$ disjoint

$f, g : \Omega \to \overline{\mathbb{R}}$ integrable functions

(a) now, as f is integrable function

$$\Rightarrow \int_\Omega f^+ d\mu < \infty, \quad \int_\Omega f^- d\mu < \infty$$

$$\Rightarrow \int_\Omega |f| d\mu < \infty$$

then by definition, $\int_A f d\mu = \int_\Omega f \chi_A d\mu$

$$|f\chi_A| = |f| \cdot |\chi_A| \leq |f|$$

$$\Rightarrow \int_\Omega |f||\chi_A| d\mu \leq \int_\Omega |f| d\mu < \infty$$

$$\Rightarrow \left| \int_\Omega f\chi_A d\mu \right| \leq \int_\Omega |f\chi_A| d\mu < \infty$$

$$\Rightarrow \int_A f d\mu = \int_\Omega f\chi_A d\mu < \infty$$

(b) $\int_\Omega (f+g) d\mu = \int_\Omega f d\mu + \int_\Omega g d\mu$

then $f+g$ well defined

$$|f+g| \leq |f| + |g|$$

$$\Rightarrow \int_\Omega |f+g| d\mu \leq \int_\Omega |f| d\mu + \int_\Omega |g| d\mu < \infty$$

$$\Rightarrow \int_\Omega |f+g| d\mu < \infty$$

$$\Rightarrow \left| \int_\Omega f+g \, d\mu \right| < \infty$$

$\Rightarrow f+g$ is integrable

$h = f+g \quad h^+ = \max(h, 0)$
$h^- = -\min(h, 0)$

$h = h^+ - h^-$
$f = f^+ - f^-$
$g = g^+ - g^-$

$$f^+ - f^- + g^+ - g^- = (f+g)^+ - (f+g)^-$$

$$\Rightarrow (f^+ + g^+) + (f+g)^- = f^- + g^- + (f+g)^+$$

$$\Rightarrow \int_\Omega f^+ d\mu + \int_\Omega g^+ d\mu + \int_\Omega (f+g)^- d\mu = \int_\Omega f^- d\mu + \int_\Omega g^- d\mu + \int_\Omega (f+g)^+ d\mu$$

as all finite

$$\Rightarrow \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu + \int_\Omega g^+ d\mu - \int_\Omega g^- d\mu$$

$$= \int_\Omega (f+g)^+ d\mu - \int_\Omega (f+g)^- d\mu$$

$$\Rightarrow \int_\Omega (f+g)\, d\mu = \int_\Omega f\, d\mu + \int_\Omega g\, d\mu$$

(c) $\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B g d\mu \quad A, B \in \mathcal{A} \quad A \cap B = \phi$

$\chi_{A\cup B} = \chi_A + \chi_B$ as :

Case I: $x \in A,\ x \notin B \quad \chi_A = 1 \quad \chi_B = 0 \quad \chi_{A\cup B} = 1$

$1 + 0 = 1$

$\Rightarrow \chi_A + \chi_B = \chi_{A\cup B}$ or vica-versa case same

Case II: $x \notin A \cup B$ then trivial

$$\Rightarrow \int_{A\cup B} f d\mu = \int_\Omega f \chi_{A\cup B} d\mu$$

$$= \int_\Omega f(\chi_A + \chi_B)\, d\mu$$

$$= \int_\Omega f\chi_A d\mu + \int_\Omega f\chi_B d\mu$$

$$= \int_A f d\mu + \int_B f d\mu$$

(a) f is finite a.e as if not then $\exists E \in \mathcal{A}$ $\mu(E) > 0$ for all $n \in \mathbb{Z}_+$

$|f| \geq n.\chi_E$

$$\Rightarrow \int_\Omega |f| d\mu \geq n \int_\Omega \chi_E d\mu = n\mu(E)$$

as $n \to \infty$

$\int_\Omega |f| d\mu = \infty$ which contradicts

(e) $|f|$ is integrable and $\left|\int_\Omega f d\mu\right| \leq \int_\Omega |f| d\mu$

$|f| = f^+ + f^-$

$$\Rightarrow \int_\Omega |f| d\mu = \int_\Omega f^+ d\mu + \int_\Omega f^- d\mu < \infty$$

$\Rightarrow |f|$ integrable

also, $\left|\int_\Omega f\,d\mu\right| = \left|\int_\Omega f^+\,d\mu - \int_\Omega f^-\,d\mu\right|$

$\le \left|\int_\Omega f^+\,d\mu\right| + \left|\int_\Omega f^-\,d\mu\right| = \int_\Omega f^+\,d\mu + \int_\Omega f^-\,d\mu$

$= \int_\Omega |f|\,d\mu$

(f) $\forall c \in \mathbb{R}$, cf is integrable and $\int_\Omega cf\,d\mu = c\int_\Omega f\,d\mu$

case I: $c=0$, then this is trivial as

$$\int_\Omega 0\,d\mu = 0 = 0\int_\Omega f\,d\mu$$

case II: $c>0$:

$(cf)^+ = c(f^+)$

$(cf)^- = c(f^-)$

$$\int_\Omega |cf|\,d\mu = \int_\Omega (cf)^+\,d\mu + \int_\Omega (cf)^-\,d\mu$$

$$\le \underbrace{c\left|\int_\Omega f^+\,d\mu\right| + c\left|\int_\Omega f^-\,d\mu\right|}_{c\int_\Omega |f|\,d\mu < \infty} < \infty$$

$\Rightarrow cf$ is integrable

$$\int_\Omega cf\,d\mu = \int_\Omega (cf)^+\,d\mu - \int_\Omega (cf)^-\,d\mu$$

$$= c\int_\Omega f^+\,d\mu - c\int_\Omega f^-\,d\mu$$

(as we can keep adding c times; ← from below)

$$= c\int_\Omega f\,d\mu$$

case III: $c<0$: $(cf)^+ = (-c)f^-$

$(cf)^- = (-c)f^+$

$-c>0 \Rightarrow \int_\Omega |cf|\,d\mu < \infty$

$$\int_\Omega cf\,d\mu = \int_\Omega (cf)^+\,d\mu - \int_\Omega (cf)^-\,d\mu$$

$$= -c\left(\int_\Omega f^-\,d\mu - \int_\Omega f^+\,d\mu\right)$$

(← from below)

$$= c\int_\Omega f\,d\mu$$

for $c\int f^+ = \int cf^+$ also for $c \in \mathbb{N}$ $c\int_\Omega f\,d\mu = \int_\Omega cf\,d\mu$ ($\because$ addition)

$\forall c \in \mathbb{Q}_{\geq 0} \Rightarrow c\int_\Omega f\,d\mu = \int_\Omega cf\,d\mu$

$\forall c \in \mathbb{R}_{\geq 0} \Rightarrow c\int_\Omega f\,d\mu = \int_\Omega cf\,d\mu$

then-c case follows

(g) $f \geq g$ a.e then $\int_\Omega f\,d\mu \geq \int_\Omega g\,d\mu$

as $h = f - g \geq 0$ a.e

$\Rightarrow \int_\Omega h\,d\mu \geq 0 \Rightarrow \int_\Omega (f-g)\,d\mu \geq 0$

$\Rightarrow \int_\Omega f\,d\mu \geq \int_\Omega g\,d\mu$

(h) if $f \geq 0$ with $\int_\Omega f\,d\mu = 0$ then $f = 0$ a.e

$A = \{\omega \in \Omega \mid f(\omega) > 0\}$

then $\mu(A) = 0$ as pf

$$A_n = \left\{\omega \in \Omega \mid f(\omega) \geq \frac{1}{n}\right\}$$

$$A = \bigcup_{n=1}^{\infty} A_n$$

$f \geq \frac{1}{n}\chi_{A_n}$ for all fixed n

$\Rightarrow \int_\Omega f\,d\mu \geq \frac{1}{n}\mu(A_n)$

$\Rightarrow 0 \geq \frac{1}{n}\mu(A_n)$

$\Rightarrow \mu(A_n) = 0$

$\Rightarrow \mu(A) = 0$

$\Rightarrow f = 0$ a.e

(i) $f = g$ a.e then $\int_\Omega f\,d\mu = \int_\Omega g\,d\mu$

pf $f \geq g$ a.e then

$$\int_\Omega f\,d\mu \geq \int_\Omega g\,d\mu \quad \text{— ①}$$

$f \leq g$ a.e then

$$\int_\Omega f\,d\mu \leq \int_\Omega g\,d\mu \quad \text{— ②}$$

from ① ② $\int_\Omega f\,d\mu = \int_\Omega g\,d\mu$

(j) $h: \Omega \to \bar{\mathbb{R}}$ is measurable and $|h| \leq f$ then h is integrable

as $|h(\omega)| \leq f(\omega) \ \forall \omega \in \Omega$ and $|h| \geq 0$

$\Rightarrow f(\omega) \geq 0$ and given f is integrable

$\Rightarrow \int_{\Omega} f \, d\mu < \infty$

(ii) and f non-neg measurable fution

and $\int_{\Omega} |h| \, d\mu \leq \int_{\Omega} f \, d\mu < \infty$

$\Rightarrow h$ is integrable

Tutorial-8 :

$\Omega \neq \phi$, $(\Omega, \mathcal{F}, \mu)$ is measurable space

$f: \Omega \to \mathbb{C}$ is said to be measurable if both $Re(f)$ and $Im(f)$ are measurable functions

and $\int_\Omega f d\mu = \int_\Omega Re(f) d\mu + i \int_\Omega Im(f) d\mu$

1. $f, g: \Omega \to \mathbb{C}$ integrable then

$$\int_\Omega (f + cg) d\mu = \int_\Omega f d\mu + c \int_\Omega g d\mu \quad \forall c \in \mathbb{C}$$

as, $f = u_1 + i v_1$, $g = u_2 + i v_2$ $\Big\}$ $\to u_1, u_2, v_1, v_2$ are real valued integrable

$c = a + ib$ $a, b \in \mathbb{R}$

$$f + cg = (u_1 + iv_1) + (u_2 + iv_2)(a + ib)$$

$$= u_1 + iv_1 + au_2 + iav_2 + ibu_2 - bv_2$$

$$\int_\Omega f + cg \, d\mu = \int_\Omega (u_1 + au_2 - bv_2) d\mu + i \int_\Omega (v_1 + av_2 + bu_2) d\mu$$

← linearity from previous tutorial

$$= \left(\int_\Omega u_1 d\mu + i \int_\Omega v_1 d\mu \right) + a \left(\int_\Omega u_2 d\mu + i \int_\Omega v_2 d\mu \right) + ib \left(-\frac{1}{i} \int_\Omega v_2 d\mu + \int_\Omega u_2 d\mu \right)$$

$$\int_\Omega u_2 d\mu + i \int_\Omega v_2 d\mu$$

$$= \int_\Omega f \, d\mu + (a + ib) \int_\Omega g \, d\mu$$

$$= \int_\Omega f d\mu + c \int_\Omega g d\mu$$

2. $f: \Omega \to \mathbb{C}$ is integrable iff $|f|: \Omega \to [0, \infty)$ is integrable

($\Rightarrow$) $f: \Omega \to \mathbb{C}$ is integrable then

$Re(f)$ and $Im(f)$ is also integrable

i.e $\int_\Omega |Re(f)| d\mu < \infty$ & $\int_\Omega |Im(f)| d\mu < \infty$

$$|f| = \sqrt{Re^2(f) + Im^2(f)} \leq |Re(f)| + |Im(f)|$$

$$\Rightarrow \int_\Omega |f|\,d\mu \le \int_\Omega |\mathrm{Re}(f)|\,d\mu + \int_\Omega |\mathrm{Im}f|\,d\mu < \infty$$

$\therefore$ $|f|$ is integrable

($\Leftarrow$) If $\int_\Omega |f|\,d\mu < \infty$ then as

$$|\mathrm{Re}(f)| \le |f|$$

$$|\mathrm{Im}(f)| \le |f|$$

$$\Rightarrow \int_\Omega |\mathrm{Re}(f)|\,d\mu < \infty, \quad \int_\Omega |\mathrm{Im}(f)|\,d\mu < \infty$$

$\Rightarrow \mathrm{Re}(f), \mathrm{Im}(f)$ are integrable

$\Rightarrow f$ is integrable

3. If $f: \Omega \to \mathbb{C}$ is integrable then $\left|\int_\Omega f\,d\mu\right| \le \int_\Omega |f|\,d\mu$

Let $z = \int_\Omega f\,d\mu \in \mathbb{C}$ then using euler's formula

$z = re^{i\theta}$ $\quad r = \left|\int_\Omega f\,d\mu\right|$ and $\theta \in \mathbb{R}$

$$r = \mathrm{Re}(e^{-i\theta} z) = \mathrm{Re}\left(e^{-i\theta} \int_\Omega f\,d\mu\right) = \mathrm{Re}\left(\int_\Omega e^{-i\theta} f\,d\mu\right)$$

($\because$ previous part)

$$\Rightarrow r = \int_\Omega \mathrm{Re}(e^{-i\theta} f)\,d\mu$$

Now $\mathrm{Re}(w) \le |w| \ \forall\, w \in \mathbb{C}$

$$\Rightarrow \mathrm{Re}(e^{-i\theta} f) \le |e^{-i\theta} f| = |e^{-i\theta}||f| = |f|$$

$$\Rightarrow r \le \int_\Omega |f|\,d\mu$$

$$\Rightarrow \left|\int_\Omega f\,d\mu\right| \le \int_\Omega |f|\,d\mu < \infty$$

4. $\{f_n\}_{n\in\mathbb{N}}$ seq of complex measurable fuctions

$$f(w) = \lim_{n\to\infty} f_n(w)$$

if $\exists$ integrable fuction $g: \Omega \to [0,\infty]$ s.t

$|f_n(w)| \le g(w) \quad \forall\, w \in \Omega$

then f_n, f are integrable

with

$$\lim_{n\to\infty}\int_\Omega |f_n - f|\,d\mu = 0 \Rightarrow \lim_{n\to\infty}\int_\Omega f_n\,d\mu = \int_\Omega f\,d\mu$$

let $f_n = u_n + i v_n$

$f = u + iv$

$f_n \to f$ pointwise $\Rightarrow |f_n - f| \to 0$ as $n\to\infty$

and $u_n \to u$, $v_n \to v$ pointwise

then $|f_n - f| \le |f_n| + |f|$

$|f_n(\omega)| \le g(\omega)\ \forall \omega \in \Omega$

$\Rightarrow |f(\omega)| \le g(\omega)\ \forall \omega \in \Omega$

as $n \to \infty$

$\Rightarrow |f_n(\omega)| + |f(\omega)| \le 2g(\omega)\ \forall \omega \in \Omega$

$\Rightarrow |f_n(\omega) - f(\omega)| \le 2g(\omega)\ \forall \omega \in \Omega$

were given g is integrable

$g : \Omega \to [0,\infty]$

$\Rightarrow 2g$ integrable and

$2g : \Omega \to [0,\infty]$

let $h_n(\omega) = |f_n(\omega) - f(\omega)|$

then $h_n : \Omega \to [0,\infty]$

and $\lim_{n\to\infty} h_n(\omega) = 0\ \forall \omega \in \Omega$

and $|h_n(\omega)| \le 2g(\omega)\ \forall \omega \in \Omega$

by Dominated convergence theorem:

0 is integrable (trivial) and

$\lim_{n\to\infty}\int_\Omega h_n\,d\mu = 0$ and $\lim_{n\to\infty}\int_\Omega h_n\,d\mu = 0$

$\Rightarrow \lim_{n\to\infty}\int_\Omega |f_n - f|\,d\mu = 0$

$\Rightarrow \lim_{n\to\infty}\left|\int_\Omega f_n\,d\mu - \int_\Omega f\,d\mu\right| \le \lim_{n\to\infty}\int_\Omega |f_n - f|\,d\mu = 0$

$\Rightarrow \lim_{n\to\infty}\int_\Omega f_n\,d\mu = \int_\Omega f\,d\mu$

now as $|f_n(\omega)| \leq g(\omega)$

$$\int_\Omega |f_n| d\mu < \infty \quad (\because g \text{ is integrable})$$

also, $\left|\int_\Omega f d\mu\right| \leq \lim_{n\to\infty} \int_\Omega |f_n| d\mu < \infty$

$$\Rightarrow \left|\int_\Omega f d\mu\right| < \infty$$

$\Rightarrow f$ is integrable

5. $f: \Omega \to \mathbb{C}$ measurable

then $f = (u^+ - u^-) + i(v^+ - v^-)$

$\exists \{\psi_{i,n}\}_{n=1}^\infty$ basic s.t $\psi_{i,n} \uparrow \psi_i$ ($\because$ done in class for non-neg measurable)

$\psi_1 = u^+ \quad \psi_2 = u^- \quad \psi_3 = v^+ \quad \psi_4 = v^-$

$\{\psi_n\}_{n=1}^\infty$ s.t $\lim_{n\to\infty} \psi_n = f \qquad \psi_n = \psi_{1,n} - \psi_{2,n} + i(\psi_{3,n} - \psi_{4,n})$

also, by making partitions equal $\Rightarrow \{\psi_n\}_{n=1}^\infty$ is complex basic

$$|\psi_n|^2 = (\psi_{1,n} - \psi_{2,n})^2 + (\psi_{3,n} - \psi_{4,n})^2$$

$$|Re(\psi_n)| = \psi_{1,n} + \psi_{2,n}$$

$$\leq \psi_{1,n+1} + \psi_{2,n+1} = |Re(\psi_{n+1})|$$

$$\Rightarrow |Re[\psi_n)| \leq |Re(\psi_{n+1})| \leq \psi_1 + \psi_2 = |Re(f)|$$

similarly $|Im(\psi_n)| \leq |Im(\psi_{n+1})| \leq \psi_3 + \psi_4 = |Im(f)|$

$$\Rightarrow Re^2(\psi_n) \leq Re^2(\psi_{n+1}) \leq Re^2(f)$$

$$Im^2(\psi_n) \leq Im^2(\psi_{n+1}) \leq Im^2(f)$$

$$\Rightarrow |\psi_n|^2 \leq |\psi_{n+1}|^2 \leq |f|^2$$

$$\Rightarrow |\psi_n| \leq |\psi_{n+1}| \leq |f|$$

$$\Rightarrow 0 \leq |\psi_1| \leq |\psi_2| \leq \cdots \leq |f|$$

Tutorial-9 :

1. $L^*(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$

$f = \chi_{[0,1]} \quad g = \chi_{[2,3]}$

$\|f\|_p^p = \int_{\mathbb{R}} |\chi_{[0,1]}|^p d\lambda$

$= 1$

$\Rightarrow \|f\|_p = 1$

similarly $\|g\|_p = 1$

$$|f+g| = \begin{cases} 1 & ; [0,1] \\ 0 & ; [2,3] \end{cases}$$

$$\|f+g\|_p = \left(\int_{\mathbb{R}} |f+g|^p d\lambda\right)^{\frac{1}{p}}$$

$$= (1+1)^{\frac{1}{p}}$$

$$= 2^{1/p}$$

$\|f+g\|_p \leq \|f\|_p + \|g\|_p$

$2^{1/p} \leq 1+1 = 2$

$2^{1/p} \leq 2^1 \Rightarrow \frac{1}{p} \leq 1 \Rightarrow p \geq 1$

3. $f, g \in L^1(\mathbb{R}) \quad f, g \in L^1(\Omega)$

$((0,1), \mathcal{B}(0,1), \lambda)$

$f = \frac{1}{\sqrt{x}} \Rightarrow \int_0^1 f\,dx < \infty$ as $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x}$

$g = \frac{1}{\sqrt{x}} \Rightarrow \int_0^1 g\,dx < \infty$

but $fg = \frac{1}{x} \quad \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^1 \frac{1}{x} dx \to \infty$

2. $l^p(\mathbb{C}) = \left\{ \{x_n\}_{n=1}^{\infty} \subset \mathbb{C} \;\middle|\; \sum_i |x_i|^p < \infty \right\}$

$\{f_n\}_{n=1}^{\infty}$ is cauchy in $l^p(\mathbb{N})$

$\Rightarrow \{f_n(k)\}_{k=1}^{\infty}$ is conv in $\mathbb{C} \quad \forall i = 1$

$$|f_n(k_0) - f_m(k_0)|^p \leq \sum_{j \geq i}^{\infty} |f_n(j) - f_m(j)|^p < \varepsilon^p \quad \forall i = 1$$

$\Rightarrow |f_n(k_0) - f_m(k_0)|^p < \varepsilon^p$

as $\|f_n - f_m\|_p < \varepsilon \Rightarrow \|f_n - f_m\|^p < \varepsilon^p \Rightarrow \sum_{j \geq i} |f_n(j) - f_m(j)|^p \leq \varepsilon^p$

$\Rightarrow |f_n(k_0) - f_m(k_0)| < \varepsilon \Rightarrow \langle f_m(k_0) \rangle$ cauchy $\Rightarrow f_m(k_0)$ converges

$f(k) = \lim_{n\to\infty} f_n(k) \quad \forall k \in \mathbb{N}$

$\langle f_n \rangle$ is bounded as $\forall n$, $\exists M > 0$ s.t

$\|f_n\|_p \le M \ \forall n \in \mathbb{N}$

$\Rightarrow \sum_{k\ge 1} |f_n(k)|^p \le M^p \ \forall n \in \mathbb{N}$ ——①

$g_n(k) := |f_n(k)|^p \qquad f_n(k) \to f(k)$

$\Rightarrow \lim_{n\to\infty} g_n(k) = |f(k)|^p \qquad \Rightarrow |f_n(k)| \to |f(k)|$

$\Rightarrow |f_n(k)|^p \to |f(k)|^p$

by fatou's lemma:

$\int \lim_{n\to\infty} g_n(k) \le \lim_{n\to\infty} \int g_n(k)$

$\Rightarrow \int |f(k)|^p \, d\mu \le \lim_{n\to\infty} \int g_n(k) \, d\mu$

$\Rightarrow \sum_{k\ge 1} |f(k)|^p \le \lim_{n\to\infty} \sum_{k\ge 1} |f_n(k)|^p \le M^p \ (\because ①)$

$\Rightarrow \sum_{k\ge 1} |f(k)|^p < \infty$

$\Rightarrow f \in \ell^p$

now $\|f_n - f\|_p \to 0$ as $n \to \infty$

$\|f_n - f\|_p^p = \left| \sum_{k\ge 1} |f_n(k) - f(k)|^p \right|$ $\qquad x_n \to x$

now, $|f_n(k) - f(k)|$ $\qquad \lim_{n\to\infty} (\alpha - x_n) \to \alpha - x$

$= \lim_{m\to\infty} |f_n(k) - f_m(k)|$ $\qquad \lim_{n\to\infty} (x_m - x_n) \to x_m - x$

$h_m(k) = |f_n(k) - f_m(k)|^p$

fatou's lemma $\qquad \sum_{m=1}^{\infty} |f_n(k) - f(k)|^p \le \lim_{m\to\infty} \left(\sum_{m=1}^{\infty} |f_m(k) - f_n(k)|^p \right)$

as $n, m \to \infty$

$\Rightarrow \|f_n - f\|_p \to 0$

4. f, g on $\mathbb{R}^n$

$f * g(x) := \int f(x-y) g(y) \, d\lambda_n(y) \quad x \in \mathbb{R}^n$

$f, g \in L^1(\mathbb{R}^n)$ $\quad f * g$ i.e $\int_{\mathbb{R}^n} |f * g| \, dx < \infty$ i.e $\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y) g(y) \, dy \right| dx < \infty$

(1) we have to show

$$\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y)g(y)dy \right| dx < \infty$$

now $|f*g(x)| = \left| \int_{\mathbb{R}^n} f(x-y)g(y)\,dy \right| \le \int_{\mathbb{R}^n} |f(x-y)||g(y)|dy$

$$\Rightarrow \int |f*g(x)| \le \iint_{\Omega_1 \Omega_2} \left(\int |f(x-y)||g(y)|dy \right) dx$$

Tonelli's theorem $\int_{\Omega_1} \left(\int_{\Omega_2} |f(x-y)||g(y)|dy \right) dx$

$$= \int_{\Omega_2} \int_{\Omega_1} |g(y)||f(x-y)|\,dx\,dy$$

$$= \int_{\Omega_2} |g(y)| \left(\int_{\Omega_1} |f(x-y)|dx \right) dy$$

$x-y=w$

$$= \int_{\Omega_2} |g(y)| \left(\int_{\Omega_1} |f(w)|dw \right) dy$$

$$= \|f\|_1 \|g\|_1$$

$\Rightarrow f*g \in L^1(\mathbb{R}^n)$

$$(f*g)*h(x) = \int (f*g)(x-y)\,h(y)\,d\lambda^n y$$

$$= \int \left[\int f(x-y-z)\,g(z)\,d\lambda^n z \right] h(y)\,d\lambda^n y$$

$z = w-y$, $dz = dw$

$$= \int \int f(x-w)g(w-y)\,d\lambda^n w\,h(y)\,d\lambda^n y$$

$$= \int \int f(x-w)g(w-y)h(y)\,d^n y\,d^n w$$

($\because$ fubini)

$$= \int f(x-w)\,g*h(w)\,d^n w$$

$$= f*(g*h)(x)$$

1. $L^p(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ $0<p<1$

$f = \chi_{[0,1]}$ $g = \chi_{[2,3]}$

now, $\|f\|_p^p = \int_{\mathbb{R}} |\chi_{[0,1]}|^p d\lambda$

$= 1$

$\|g\|_p^p = 1$

$\Rightarrow f, g \in L^p(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$

$$|f+g| = \begin{cases} 1 ; [0,1] \\ 1 ; [2,3] \\ 0 ; \text{otherwise} \end{cases}$$

$\|f+g\|_p^p = \int_{\mathbb{R}} |f(\omega) + g(\omega)|^p d\lambda$

$= 1+1 = 2$

$\Rightarrow \|f+g\|_p = (2)^{1/p}$

now

$\|f\|_p + \|g\|_p = 2$

as $p \in (0,1) \Rightarrow 1/p \in (1,\infty)$

$2 < 2^{1/p}$

$\Rightarrow \|f\|_p + \|g\|_p < \|f+g\|_p$

or minkowski's inequality does not hold

3. $fg \in L^1(\Omega)$ when $f, g \in L^1(\Omega)$

let $((0,1), \mathcal{B}(0,1), \lambda)$

$f = \frac{1}{\sqrt{x}} \Rightarrow \int_0^1 f(x)\,dx = 2 < \infty \Rightarrow f \in L^1((0,1))$

$g = \frac{1}{\sqrt{x}} \Rightarrow g \in L^1((0,1))$

but $f \cdot g = \frac{1}{x}$ $\int_0^1 \frac{1}{x} dx \to \infty$ as $\lim_{\varepsilon \to 0+} \int_\varepsilon^1 \frac{1}{x} dx \to \infty$

so, $fg \notin L^1((0,1))$

2. $\Omega = \mathbb{N}$ $\mathcal{A} = \mathcal{P}(\mathbb{N})$ μ = counting measure

$f : \mathbb{N} \to \mathbb{C}$

$\{x_n\}_{n=1}^\infty \subseteq \mathbb{C}$ $f \leftrightarrow \{f(n)\}_{n=1}^\infty$

$p \in [1,\infty)$ $L^p(\Omega) = \{\{x_n\}_{n=1}^\infty \subset \mathbb{C} \mid \sum_i |x_i|^p < \infty\}$

$\downarrow$

$\ell^p(\mathbb{N})$ or ℓ^p

$\{f_n\}_{n=1}^{\infty}$ cauchy in $\ell^p(\mathbb{N})$ then we have to show

$\forall k \in \mathbb{N}$ $\{f_n(k)\}_{n=1}^{\infty}$ is conv in $\mathbb{C}$

$f : \mathbb{N} \to \mathbb{C}$ $\quad f(k) = \lim_{n\to\infty} f_n(k)$ $\forall k \in \mathbb{N}$

$f \in \ell^p(\mathbb{N})$

and $\|f_n - f\|_p \to 0$ as $n \to \infty$

now, $|f_n(k) - f_m(k)|^p$

$\leq \sum_{j\geq 1}^{\infty} |f_n(j) - f_m(k)|^p < \varepsilon^p$

but as $\{f_n\}_{n=1}^{\infty}$ is cauchy seq

$\|f_n - f_m\|_p^p < \varepsilon^p$

$\Rightarrow |f_n(k) - f_m(k)| < \varepsilon^p$

$\Rightarrow \{f_n(k)\}_{n=1}^{\infty}$ cauchy $\forall k \in \mathbb{N}$

$\Rightarrow \{f_n(k)\}_{n=1}^{\infty}$ conv ($\because \mathbb{C}$ is complete)

now, let $f(k) = \lim_{n\to\infty} f_n(k)$ $\forall k \in \mathbb{N}$

$\{\|f_n\|_p\}_{n=1}^{\infty}$ is bounded ($\because \{f_n\}_{n=1}^{\infty} \subseteq \ell^p(\mathbb{N}) \Rightarrow$ all are bounded)

$\exists m > 0$ s.t $\quad \exists m_n > 0$, $\|f_n\|_p \leq m_n$, $m = \max$ of all m

$\|f_n\|_p \leq m$ $\forall n \in \mathbb{N}$

$\Rightarrow \sum_{k\geq 1} |f_n(k)|^p \leq m^p$ $\forall n \in \mathbb{N}$ —— ①

also, $g_n(k) = |f_n(k)|^p$ $\quad f_n(k) \to f(k)$

$\Rightarrow \lim_{n\to\infty} g_n(k) = |f(k)|^p$

by fatou's lemma:

$$\int \lim_{n\to\infty} g_n(k) \leq \lim_{n\to\infty} \int g_n(k)$$

$$\Rightarrow \sum_{k\geq 1} |f(k)|^p \leq \lim_{n\to\infty} \sum_{k\geq 1} |f_n(k)|^p \leq m^p \quad (\because ①)$$

$$\Rightarrow \sum_{k\geq 1} |f(k)|^p < \infty$$

$\Rightarrow f \in \ell^p$

now, $\|f_n - f\|_p \to 0$ as $n \to \infty$ as

$$\|f_n - f\|_p^p = \left| \sum_{k \geq 1} |f_n(k) - f(k)|^p \right|$$

$$\underbrace{|f_n(k) - f(k)|}_{\text{raise to } p} = \lim_{m \to \infty} \underbrace{|f_n(k) - f_m(k)|}_{\text{raise to } p}$$

$h(k)$ $\qquad$ $h_m(k)$

by fatou's lemma:

$$\int \lim_{n \to \infty} h_m(k) \leq \lim_{n \to \infty} \int h_m(k)$$

$$\Rightarrow \underbrace{\sum_{k=1}^{\infty} |f_n(k) - f(k)|^p}_{\|f_n - f\|_p^p} \leq \lim_{n \to \infty} \left(\underbrace{\sum_{k=1}^{\infty} |f_m(k) - f_n(k)|^p}_{\to 0} \right) \quad \text{as } n, m \to \infty$$

$$\Rightarrow \|f_n - f\|_p \to 0$$

4. f, g measurable on $\mathbb{R}^n$

$$f * g(x) := \int f(x-y) g(y) \, d\lambda_n(y)$$

$f, g \in L^1(\mathbb{R}^n)$

then $|f * g(x)| = \left| \int_{\mathbb{R}^n} f(x-y) g(y) dy \right|$

$$\leq \int_{\mathbb{R}^n} |f(x-y)||g(y)| dy$$

$$\Rightarrow \int |f * g(x)| \leq \int_{\Omega_1} \int_{\Omega_2} |f(x-y)||g(y)| \, dy \, dx$$

$$= \int_{\Omega_2} \int_{\Omega_1} |g(y)| |f(x-y)| \, dx \, dy$$

(∵ tonelli theorem as f, g are non-neg mes)

$$= \int_{\Omega_2} |g(y)| \left(\int_{\Omega_1} f(w) dw \right) dy$$

$$= \|f\|_1 \|g\|_1$$

$$\Rightarrow f * g \in L^1(\mathbb{R}^n)$$

also, $(f * g) * h$, $f * (g * h) \in L^1(\mathbb{R}^n)$ so integrable

and, $(f * g) * h(x) = \int (f * g)(x-y) h(y) d_{\lambda_n} y$

$$= \int \left[\int f(x-w) g(w-y) d\lambda_n w \right] h(y) d\lambda_n(y)$$

$z = w - z \quad dw = dz$

$$= \int\int f(x-w)\, g(w-y)\, h(y)\, d_n y\, d_n w$$

$$(\because \text{fubini})$$

$$= \int f(x-w)\, g*h(w)\, d_n(w)$$

$$= f*(g*h)(x)$$

1. $(\Omega, \mathcal{H}, \mu)$ $f: \Omega \to [0,\infty]$ measurable function

$$\mu_f(E) = \int_E f\, d\mu \quad E \in \mathcal{H}$$

① $\mu_f(\phi) = \int_\Omega f \chi_\phi\, d\mu = \int_\Omega 0\, d\mu = 0$

② let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ $E_i \cap E_j = \phi$ for $i \neq j$

$E = \bigcup_{i=1}^{\infty} E_i \in \mathcal{H}$ s.t

$$\mu_f(E) = \int_\Omega f\, \chi_E\, d\mu$$

now, $\chi_E(w) = \bigcup_{i=1}^{\infty} \chi_{E_i}(w)\ \forall w \in \Omega$

as, Case 1: $w \in E$

then, $\exists\, i \in \mathbb{N}$ unique s.t

$w \in E_i$ as $E_i \cap E_j = \phi\ \forall\, i \neq j$

so, $\chi_E(w) = 1 = \chi_{E_i}(w)$

$\chi_{E_j}(w) = 0\ \forall\, j \neq i$

$$\Rightarrow \chi_E(w) = \sum_{i=1}^{\infty} \chi_{E_i}(w)$$

also if $w \notin E$ then

$w \notin E_i\ \forall\ i \in \mathbb{N}$

$$\Rightarrow \chi_E(w) = 0 = \sum_{i=1}^{\infty} \chi_{E_i}(w)$$

so, $\mu_f(E) = \int_\Omega f\, \chi_E\, d\mu = \sum_{i=1}^{\infty} \int_\Omega f\, \chi_{E_i}\, d\mu = \sum_{i=1}^{\infty} \mu_f(E_i)$ $\left(\because \int_E f\, d\mu \in [0,\infty]\right)$

③ $\mu_f: \mathcal{H} \to [0,\infty]$ $(\because f: \Omega \to [0,\infty]$ and $\mu: \mathcal{H} \to [0,\infty])$

from ①, ②, ③ μ_f is a measure

2. $(\Omega, \mathcal{H}, \mu)$ measure space

(a) $f: \Omega \to \mathbb{R}$ measurable function

$$\nu(E) = \int_E f\,d\mu \quad E \in \mathcal{H}$$

$$\int_E f^+ d\mu = \infty \qquad \int_E f^- d\mu = \infty$$

Let $f(x) = \sin x \quad \Omega = \mathbb{R} \quad \mathcal{H} = \mathcal{B}(\mathbb{R}) \quad \mu = \lambda$

then f is measurable in both as well

also, $\int_{\mathbb{R}} f^+(x)\,dx$

$$f^+(x) = \begin{cases} \sin(x) & ; x \in [2n\pi, 2n\pi+\pi] \\ 0 & ; x \in (2n\pi+\pi, 2n\pi+2\pi) \end{cases}$$

$$\Rightarrow \int_{[2n\pi, 2n\pi+\pi]} \sin(x)\,dx = 1$$

$$\Rightarrow \int_{\mathbb{R}} f^+(x)\,dx = \infty$$

similarly $\int_{\mathbb{R}} f^-\,dx = \infty$

so, $\nu^+(E) = \int_E f^+\,dx \qquad \nu^-(E) = \int_E f^-\,dx$

s.t $\nu(E) = \nu^+(E) - \nu^-(E)$

$\nu(\mathbb{R}) = \infty - \infty$ not defined

so, μ_f not signed measure

(b) $f \in L^1(\Omega) \quad \mu_f(E) = \int_E f\,d\mu \quad E \in \mathcal{H}$

as $f \in L^1(\Omega) \Rightarrow \int_\Omega |f|\,d\mu < \infty$

① $\mu_f(\phi) = \int_\phi f\,d\mu = \int_\Omega 0\,d\mu = 0$

② for, $E = \bigcup_{i=1}^{\infty} E_i \quad \{E_i\}_{i=1}^{\infty} \subset \mathcal{H} \quad E_i \cap E_j = \phi \quad i \neq j$

$$\mu_f(E) = \int_E f\,d\mu = \int_\Omega f \chi_E\,d\mu = \sum_{i=1}^{\infty} \int_{E_i} f\,d\mu \qquad \left(\because \int_{E_i} f\,d\mu < \infty\right)$$

so from ①, ②, μ_f is a signed measure

(c) false as $\mu_1 = \lambda, \mu_2 = C$ then $\mu(\mathbb{R}) = \lambda(\mathbb{R}) - C(\mathbb{R})$ not defined

3. $(\Omega, \mathcal{H})$ measurable space $\nu : \mathcal{H} \to (-\infty, \infty]$ signed measure on $(\Omega, \mathcal{H})$

(a) (P_1, N_1) s.t $P_i \cup N_i = \Omega$ $P_i \cap N_i = \phi$

now $P_1 \setminus P_2 \subseteq P_1$

$P_1 \setminus P_2 \subseteq N_2$ $(\because N_2 = P_2^c)$

$\Rightarrow P_1 \setminus P_2 \subseteq P_1 \cap N_2$

similarly $P_2 \setminus P_1 \subseteq P_2 \cap N_1$

then $E \subseteq P_1 \setminus P_2$

$\Rightarrow \nu(E) \geq 0$ $(\because E \subseteq P_1)$
$\nu(E) \leq 0$ $(\because E \subseteq N_2)$

$\Rightarrow \nu(E) = 0$

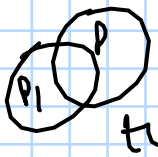
similarly, $E \subseteq P_2 \setminus P_1$, $\nu(E) = 0$

so, $E \subseteq (P_1 \setminus P_2) \cup (P_2 \setminus P_1) \Rightarrow \nu(E) = 0$

similar for $N_1 \Delta N_2$

(b) $E \cap P_2 \subseteq E \cap (P_1 \cup P_2)$

$= (E \cap (P_1)) \cup (E \cap (P_1 \Delta P_2))$



then $\nu(E \cap P_2) = \nu(E \cap P_1) + \nu(\underbrace{E \cap (P_1 \Delta P_2)}_{0})$

$\Rightarrow \nu(E \cap P_2) = \nu(E \cap P_1)$

similar for N_1, N_2

4. $(\Omega, \mathcal{A}, \mu)$ measure space

(a) $f \in L^1(\Omega) \Rightarrow f = Re(f) + i\,Im(f)$

$\mu_f(\phi) = 0$ trivial

and $Re(\nu)(E) = \sum Re(\nu)(E_i)$

$(\because$ all real values $< \infty)$

similar for img

$\Rightarrow \nu$ is sigma additive

(b) $|\mu_f|$ of μ_f

$$|\mu_f|(E) = \sup \left\{ \sum_{i=1}^{\infty} |\mu_f(E_i)| \;\middle|\; \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A},\ E_i \cap E_j = \phi,\ E = \bigcup_{i=1}^{\infty} E_i \right\}$$

$$= \sup \left\{ \sum_{i=1}^{\infty} \left| \int_{E_i} f \, d\mu \right| \right\} \leq \int_E |f| \, d\mu$$

also, $\forall \varepsilon > 0, \exists \{E_i\}_{i=1}^{\infty}$ s.t

$$\sum_{i=1}^{\infty} \left| \int_{E_i} f \, d\mu \right| > \sum_{i=1}^{\infty} \int_{E_i} |f| \, d\mu + \varepsilon$$

$$\Rightarrow \int_E |f| \, dp - \varepsilon \leq |\mu_f|(E) \leq \int_E |f| \, d\mu$$

as $\varepsilon \to 0$

$$\Rightarrow |\mu_f(E)| = \int_E |f| \, d\mu$$

(c) $(\Omega, \mathcal{H})$ complex measures $M(\Omega)$

$$\|\nu\| := |\nu|(\Omega) \quad \forall \nu \in M(\Omega)$$

$T: L^1(\Omega) \to M(\Omega)$

$T(f) = \mu_f \quad \forall f \in L^1(\Omega)$

(i) T is a linear map as $\forall f, g \in L^1(\Omega)$

then $f+g \in L^1(\Omega)$

$cf \in L^1(\Omega) \quad c \in \mathbb{C}$

$$\mu_{f+g}(E) = \int_E f+g \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu$$

$$= (\mu_f + \mu_g)(E)$$

$$\Rightarrow T(f+g) = T(f) + T(g)$$

similarly $c\mu_f(E) = \mu_{cf}(E) \Rightarrow T(cf) = cT(f)$

(ii) $\|T(f)\| = |\mu_f|(\Omega)$

$$= \int_{\Omega} |f| \, d\mu \quad (\because \text{above})$$

$$= \|f\|_{L^1}$$

(iii) T is injective as

$$\mu_f = \mu_g \Rightarrow \int_E f \, d\mu = \int_E g \, d\mu \quad \forall E \in \mathcal{H}$$

$$\Rightarrow \int_E (f-g) \, d\mu = 0$$

$$\Rightarrow f = g \text{ a.e}$$

$$\Rightarrow f = g \text{ in } L^1(\Omega) \text{ as } f \sim g$$

(iv) $T(L^1(\Omega))$ is closed subspace of $M(\Omega)$

as, $T(L^1(\Omega)) = \{ M_f \mid f \in L^1(\Omega) \}$

now let

$\{M_{f_i}\}_{i=1}^{\infty} \subseteq T(L^1(\Omega))$ i.e $f_i \in L^1(\Omega)$ be convg

i.e $\exists M \in M(\Omega)$ s.t

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t $\forall n \geq N$

$\|M_{f_n} - M\| = |M_{f_n} - M|(\Omega) < \varepsilon \Rightarrow \{M_{f_i}\}_{i=1}^{\infty}$ is cauchy

now, $\{f_i\}_{i=1}^{\infty} \subseteq L^1(\Omega)$ is s.t

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t $\forall n, m \geq N$

$\|f_n - f_m\|_{L^1} < \varepsilon$

$\exists \varepsilon$ as if not, then $\exists E \in \mathcal{H}$ $\mu(E) \neq 0$ s.t

$|f_n(\omega) - f_m(\omega)| > \varepsilon \quad \forall \omega \in E$

$\Rightarrow |M_{f_n}(E) - M_{f_m}(E)| > \varepsilon$

but this is not possible

$\Rightarrow \{f_i\}_{i=1}^{\infty}$ is cauchy

$\exists f \in L^1(\Omega)$ s.t $f_i \to f$ as $i \to \infty$

$\Rightarrow M_f = M$

$\because M \in T(L^1(\Omega))$

$\because T(L^1(\Omega))$ is closed

5. ν complex measure on $(\Omega, \mathcal{H})$

$h : \Omega \to S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \}$ is $|\nu|$-measurable

$$\nu(E) = \int_E h \, d|\nu| \quad \forall E \in \mathcal{H}$$

now, $|\nu(E)| \leq |\nu|(E)$

$\Rightarrow |\nu(E)| = 0$ when $|\nu|(E) = 0$

$\Rightarrow \nu << |\nu|$

$|\nu|$ is finite & $\nu << |\nu|$ on $(\Omega, \mathcal{H})$

then $\exists h \in L^1(\Omega, \mathcal{H}, |\nu|)$ s.t $\nu(E) = \int_E h \, d|\nu| \quad \forall E \in \mathcal{H}$

also, $|\nu|(E) = \int_E |h| \, d|\nu|$ ($\because$ above exercise)

$\Rightarrow |h| = 1$ a.e

Let $\tilde{h} = \begin{cases} h; & |h|=1 \\ 1; & |h|\neq 1 \end{cases}$

then $\tilde{h}$ is $|v|$-measurable

and $\tilde{h}: \Omega \to S^1 = \{z \in \mathbb{C} \mid |z|=1\}$

Tutorial-10 :

1. $L^q(\Omega) \subseteq L^p(\Omega)$ $1 \le p < q \le \infty$ and $\mu(\Omega) < \infty$

case 1: $q < \infty$: let $f \in \mathcal{L}^q(\Omega) \Rightarrow \int_\Omega |f|^q d\mu < \infty$

$r = \frac{q}{p} > 1 \quad \frac{1}{r} + \frac{1}{s} = 1$

$\Rightarrow s = \frac{r}{r-1}$

then by Hölder's inequality :

$$\int_\Omega (|f|^p \cdot 1)\, d\mu \le \left(\int_\Omega |f|^{ps} d\mu\right)^{\frac{1}{r}} \left(\int_\Omega 1 \, d\mu\right)^{\frac{1}{s}}$$

$$\le \left(\int_\Omega |f|^q d\mu\right)^{p/q} \left(\mu(\Omega)\right)^{(q-p)/q} < \infty$$

case 2: $q = \infty$: $f \in \mathcal{L}^\infty(\Omega)$, then $\exists \|f\|_\infty < \infty$

i.e $|f(x)| \le \|f\|_\infty$ a.e

$$\Rightarrow \int_\Omega |f|^p d\mu \le \int_\Omega \|f\|_\infty^p d\mu = \|f\|_\infty^p \mu(\Omega) < \infty$$

for $\mu(\Omega) = \infty$ there is no relation $p <$

$L^p(\mathbb{R}) \not\subseteq L^q(\mathbb{R})$ → $\frac{1}{\sqrt{x}}$, $\frac{1}{x}$

$\not\supseteq$ → $\frac{1}{x}$, $\frac{1}{x^2}$

more over general case: (infinite $\mu(\Omega)$)

$f(x) = \frac{1}{(x)^{1/p}} \chi_{[0,1]} \notin \mathcal{L}^p(\mathbb{R})$

but $p > q$ $f \in L^q(\mathbb{R})$ NOT $L^p(\mathbb{R})$

similarly $f(x) = \frac{1}{x^{1/p}} \chi_{[1,\infty)}$ $q > p$

$$\int_{\mathbb{R}} |f|^p d\mu = \int_1^\infty \frac{1}{x} dx \Rightarrow f \notin L^p(\mathbb{R})$$

but $f \in L^q(\mathbb{R})$

2. $\Omega = \mathbb{N}$ $\mathcal{A} = \mathcal{B}(\mathbb{N})$ μ = counting measure

$\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$ $\forall\, 1 \le p < q \le \infty$

$\ell^p(\mathbb{N}) \supsetneq \ell^q(\mathbb{N})$ → as $f \in \ell^q(\mathbb{N})$

$f = \left(1, \frac{1}{2^{1/p}}, \frac{1}{3^{1/p}}, \frac{1}{4^{1/p}}, \dots\right)$

$$\|f\|_q = \left(\sum \frac{1}{(n)^{q/p}}\right)^{1/q}$$

as $q > p$ converges

$$\|f\|_p = \left(\sum \frac{1}{n}\right)^{1/p} \text{ diverges}$$

but $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$

for $f \in \ell^p(\mathbb{N})$ then

$$\sum_{i=1}^{\infty} |f(i)|^p < \infty$$

now $\exists N \in \mathbb{N}$ s.t

$$|f(n)| \in [0,1) \quad \forall n \geq N$$

$$\Rightarrow \sum_{i=N}^{\infty} |f(i)|^q < \sum_{i=N}^{\infty} |f(i)|^p < \infty$$

$$\Rightarrow f \in \ell^q(\mathbb{N})$$

also for $q = \infty$: as $\{|f(i)|^p\}_{i=1}^{\infty}$ is bounded ($\because$ conv)

$\Rightarrow \exists$ limsup

$\Rightarrow f \in \ell^\infty(\mathbb{N})$

4. f_n, f measurable on finite measure space $(\Omega, \mathcal{H}, \mu)$

let $f_n \xrightarrow{\text{pt. wise}} f$ a.e on Ω then $\forall \varepsilon > 0, \exists \Omega_\varepsilon \subset \mathcal{H}$ s.t

$$\mu(\Omega_\varepsilon^c) < \varepsilon \text{ and } f_n \xrightarrow{\text{uniform}} f \text{ on } \Omega_\varepsilon$$

now, $\Omega = \Omega_0 \cup \Omega_0^c$ where $\mu(\Omega_0^c) = 0$

given $x \in \Omega_0, \forall \varepsilon > 0, \exists N_{\varepsilon,x}$ s.t $|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N_{\varepsilon,x}$

given $\varepsilon > 0$, $\Omega_0 = \bigcup_{N \in \mathbb{N}} \Big(\underbrace{\bigcap_{n \in \mathbb{N}} \{x \mid |f_n(x) - f(x)| < \varepsilon\}}_{\Omega_{N,\varepsilon}} \Big)$

$$\Omega_{N+1,\varepsilon} \supseteq \Omega_{N,\varepsilon}$$

$$\Omega_{N+1,\varepsilon} \cap \{x \mid |f_N(x) - f(x)| < \varepsilon\}$$

$\{\Omega_{N,\varepsilon}\}_{N=1}^{\infty}$ is increasing

$\Rightarrow \Omega_0^c = \bigcap_{N \in \mathbb{N}} \Omega_{N,\varepsilon}^c$ $\quad \{\Omega_{N,\varepsilon}^c\}_{N=1}^{\infty}$ is decreasing

and as finite measure $\Omega_0^c \in \mathcal{H}$ ($\because$ continuity as finite measure)

$\Rightarrow \mu(\Omega_0^c) = \lim_{N\to\infty} \mu(\Omega_{N,\varepsilon}^c)$

and $\mu(\Omega_0^c) = 0 \Rightarrow \lim_{N\to\infty} \mu(\Omega_{N,\varepsilon}^c) = 0$

for any $\varepsilon = \frac{1}{k}$, we have $\mu(\Omega_{N,\frac{1}{k}}^c) \to 0$ as $N \to \infty$

given $\varepsilon > 0$, if we take intersection of all k, we get

1. $\mu(\Omega) < \infty$

Case I: $q < \infty$ then for any $1 \le p < q$

for $f \in L^q(\Omega) \Rightarrow \int_\Omega |f|^q d\mu < \infty$

$\frac{1}{r} + \frac{1}{s} = 1 \quad s = \frac{r}{r-1} \quad r = \frac{q}{p} > 1$

by Holder's inequality $\int_\Omega |f|^p \cdot 1 \, d\mu \le \left(\int |f|^{pr} d\mu\right)^{\frac{1}{r}} \left(\int 1 d\mu\right)^{\frac{1}{s}}$

$= \left(\int |f|^q d\mu\right)^{p/q} \left(\mu(\Omega)\right)^{\frac{q-p}{q}} < \infty$

$\Rightarrow f \in L^p(\Omega)$

Case II: $q = \infty$: $f \in L^\infty(\Omega)$

$\Rightarrow \|f\|_\infty < \infty$

i.e $|f(x)| \le \|f\|_\infty$ a.e

$\Rightarrow \int_\Omega |f|^p d\mu \le \int_\Omega \|f\|_\infty^p d\mu = \|f\|_\infty^p \mu(\Omega) < \infty$

$\Rightarrow f \in L^p(\Omega)$

so, for $\mu(\Omega) < \infty$

$1 \le p < q \le \infty$

$L^q(\Omega) \subseteq L^p(\Omega)$

2. $\Omega = \mathbb{N} \quad \mathcal{A} = \mathcal{P}(\mathbb{N}) \quad \mu =$ counting measure $\quad 1 \le p < q \le \infty$

$\ell^p(\mathbb{N}) \not\supset \ell^q(\mathbb{N})$ as

$f = \left(1, \frac{1}{2^{1/p}}, \frac{1}{3^{1/p}}, \dots\right)$

$\|f\|_p = \left(\sum \frac{1}{n}\right)^{1/p}$ diverges but $\|f\|_q = \left(\sum \frac{1}{n^{q/p}}\right)^{\frac{1}{q}}$ converges

Case I: $q < \infty$

now, for $f \in \ell^p(\mathbb{N})$

$\Rightarrow \sum_{i=1}^{\infty} |f(i)|^p < \infty$

$\Rightarrow \exists\ N \in \mathbb{N}$ s.t. $|f(n)| \in [0,1)\ \forall\ n \geq N$

$\Rightarrow |f(n)|^q \leq |f(n)|^p$

$\Rightarrow \sum_{i=N}^{\infty} |f(n)|^q \leq \sum_{i=N}^{\infty} |f(n)|^p < \infty$

$\Rightarrow \sum_{i=1}^{\infty} |f(n)|^q < \infty$

$\Rightarrow f \in \ell^q(\mathbb{N})$

case II: $q = \infty$:

as $\{|f(i)|^p\}_{i=1}^{\infty}$ is convergent to 0

$\Rightarrow \exists$ limsup

$\Rightarrow f \in \ell^\infty(\mathbb{N})$

3. $\Omega = \mathbb{R}$ $\mathcal{A} = \mathcal{B}(\mathbb{R})$ $\mu = \lambda$ lebesgue measure on $\mathbb{R}$ $p \in [1, \infty)$

let $f(x) = \frac{1}{x^{1/q}} \chi_{[0,1]}$ for $p < q$

as $\int |f|^p d\lambda = \int_0^1 x^{-p/q}\, dx$

$$= \left| \frac{x^{-p/q+1}}{1-p/q} \right|_0^1 = \frac{q}{q-p}$$

$\Rightarrow f \in L^p(\mathbb{R})$

but $\int |f|^q d\lambda = \int_0^1 \frac{1}{x}\, dx \longrightarrow \infty$

$\Rightarrow f \notin L^q(\Omega)$

and if $p > q$ then let

$$f(x) = \frac{1}{x^{1/q}} \chi_{[1,\infty)}$$

then $\int |f|^p d\lambda = \int^{\infty} x^{-p/q}\, dx$

$$= \left(\frac{x^{1-p/q}}{1-p/q} \right)_1^{\infty} = \frac{-1}{1-p/q} = \frac{-q}{q-p} = \frac{q}{p-q}$$

$\Rightarrow f \in L^p(\mathbb{R})$

but $f \notin L^q(\mathbb{R})$ as

$$\int |f|^q d\lambda = \int_1^\infty \frac{1}{x} dx \to \infty$$

4. $\mu(\Omega) < \infty$

$$\Omega_0 = \bigcup_{N \in \mathbb{N}} \left(\bigcap_{n \geq N} \underbrace{\{x \mid |f_n(x) - f(x)| < \varepsilon\}}_{\Omega_{N,\varepsilon}} \right)$$

$$\Omega_{N,\varepsilon} = \Omega_{N+1} \cap \{x \mid |f_N(x) - f(x)| < \varepsilon\}$$
$$\subseteq \Omega_{N+1}$$

also, $\mu(\Omega) < \infty$

then $\Omega^c_{N,\varepsilon} \supseteq \Omega^c_{N+1,\varepsilon}$ and

$$\Rightarrow \bigcap_{i=1}^\infty \Omega^c_{i,\varepsilon} = \Omega_0^c$$

is s.t $\mu(\Omega_0^c) = 0$

also, $\Omega_0 = \bigcup_{i=1}^\infty \Omega_{i,\varepsilon}$

$\Omega_0 \subset \Omega$ s.t
$\mu(\Omega \setminus \Omega_0) = 0 < \varepsilon$

and on Ω_0: as $\Omega_{N,\varepsilon} \subseteq \Omega_{N+1,\varepsilon}$

on Ω_0 by convention, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N, \forall x \in \Omega_0$$

Tutorial-11 :

1. given $(\Omega, \mathcal{A})$ $\nu : \mathcal{A} \to \mathbb{C}$

$$|\nu|(E) := \sup\left(\left\{ \sum_{i=1}^{\infty} |\nu(E_i)| \;\middle|\; \{E_i\}_{i=1}^{n} \subseteq \mathcal{A},\ E_i \cap E_j = \phi,\ E = \bigcup_{i=1}^{\infty} E_i \right\} \right)$$

$$\alpha_E = \sup\left(\left\{ \sum_{i=1}^{\infty} |\nu(E_i)| \;\middle|\; n \in \mathbb{N},\ E = \bigcup_{j=1}^{n} E_j,\ \{E_j\}_{i=1}^{n} \subset \mathcal{A},\ E_j \cap E_{j'} = \phi \text{ for } j \neq j' \right\} \right) \quad \text{— (2)}$$

(1), (2) $\Rightarrow |\nu|(E) \geq \alpha_E$

we have to show $\alpha_E \geq |\nu|(E)$

case 1 : $|\nu|(E) = \infty$

given $M > 0$, $\exists \{E_i\} \subseteq \mathcal{A}$ s.t

$$\lim_{n \to \infty} S_n = \sum_{i=1}^{\infty} |\nu(E_i)| > M$$

$$S_n = \sum_{i=1}^{n} |\nu(E_i)|, \quad \exists n_0 \text{ s.t } S_{n_0} > M$$

$$\Rightarrow \sum_{i=1}^{n_0} |\nu(E_i)| > M$$

$$\Rightarrow \alpha_E > M \quad \forall M$$

$$\Rightarrow \alpha_E = \infty$$

case 2 : $|\nu|(E) < \infty$

given $\varepsilon > 0$, $\{E_i\} \subseteq \mathcal{A}$ s.t

$$|\nu|(E) - \varepsilon \leq \sum_{i=1}^{\infty} |\nu(E_i)|$$

$\exists n_0$ s.t $\sum_{i=n_0}^{\infty} |\nu(E_i)| < \varepsilon$

$$|\nu|(E) - \varepsilon < \sum_{i=1}^{n_0 - 1} |\nu(E_i)| + \sum_{i=n_0}^{\infty} |\nu(E_i)|$$

$$\leq \alpha_E + \varepsilon$$

$$\Rightarrow |\nu|(E) \leq \alpha_E$$

so, $|\nu|(E) = \alpha_E$

1. $|\nu| : \mathcal{A} \to [0, \infty]$

$$|\nu|(E) \quad \alpha_E = \sup\left\{ \sum_{i=1}^{n} |\nu(E_i)| \right\}$$

case 1 : $|\nu|(E) = \infty$

then given $M > 0$, $\exists \{E_i, m\}_{i=1}^{n}$ s.t $S_n = \sum_{i=1}^{n} |\nu(E_{i,m})|$

$$\lim_{n \to \infty} S_n > M$$

and S_n is inc $\Rightarrow \exists n_0$ s.t $S_{n_0} > M$

$$\Rightarrow \sum_{i=1}^{n_0} |\nu(E_i, m)| > m$$

$$\Rightarrow \alpha_E > m \quad \forall\, m$$

$$\Rightarrow \alpha_E = \infty$$

case 2: $|\nu|(E) < \infty$

$\forall \varepsilon > 0$, $\exists\ \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$

s.t $|\nu|(E) - \varepsilon < \sum_{i=1}^{\infty} |\nu(E_i)|$

$\exists\ n_0 \in \mathbb{N}$ s.t

$$\sum_{i=n_0}^{\infty} |\nu(E_i)| < \varepsilon$$

$$\Rightarrow |\nu|(E) - \varepsilon < \varepsilon + \sum_{i=1}^{n_0+1} |\nu(E_i)|$$

$$\Rightarrow |\nu|(E) < 2\varepsilon + \alpha_E$$

$$\Rightarrow |\nu|(E) \leq \alpha_E$$

also trivially $\alpha_E \leq |\nu|(E)$

$\therefore$, $|\nu|(E) = \alpha_E$

3. $|\nu|$ is a measure on $(\mathcal{X}, \mathcal{H})$ if for $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{H}$ $A_i \cap A_j = \phi$ $i \neq j$ we have $|\nu|\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} |\nu|(A_k)$

$\nu : \mathcal{H} \longrightarrow \mathbb{C}$ is a complex measure

$|\nu| : \mathcal{H} \longrightarrow [0, \infty]$

$$|\nu|(E) = \sup \left\{ \sum_{j=1}^{\infty} |\nu(E_j)| \;\middle|\; E = \bigcup_{j=1}^{\infty} E_j,\ \{E_j\}_{j=1}^{\infty} \in \mathcal{H},\ E_j \cap E_{j'} = \phi\ j \neq j' \right\}$$

let $m \in \mathbb{N}$ $\forall\ n \in \{1, 2, \ldots m\}$

$\{E_{j,k}\}_{j=1}^{n_k} \subseteq A_k$ $\quad E_{j,k} \subseteq A_n$

$$\{E_{j,k} \mid 1 \leq k \leq m,\ 1 \leq j \leq n_k\} \subseteq \bigcup_{k=1}^{\infty} A_k$$

now, $|\nu|(A_k) = \sup \left\{ \sum_{j=1}^{n_k} |\nu(E_{j,k})| \;\middle|\; A_k = \bigcup_{j=1}^{n_k} E_{j,k} \right\}$ ($\because$ (1))

$$\Rightarrow \sum_{k=1}^{m} \sum_{j=1}^{n_k} |\nu(E_{j,k})| \leq |\nu|\left(\bigcup_{k=1}^{\infty} A_k\right)$$

$$\Rightarrow \sum_{k=1}^{m} \sup \left\{ \sum_{j=1}^{n_k} |\nu(E_{j,k})| \right\} \leq |\nu|\left(\bigcup_{k=1}^{\infty} A_k\right)$$

$$\Rightarrow \sum_{k=1}^{m} |\nu|(A_k) \leq |\nu|\left(\bigcup_{k=1}^{\infty} A_k\right)$$

$$\Rightarrow \sum_{k=1}^{\infty} |\nu|(A_k) \leq |\nu|\left(\bigcup_{k=1}^{\infty} A_k\right) \quad \text{——①}$$

now, $\{E_j\}_{j=1}^{n} \subseteq \bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$

$$\Rightarrow \sum_{k=1}^{\infty} |\nu|(A_k) \geq \sum_{k=1}^{\infty} \sum_{j=1}^{n} |\nu(E_j \cap A_k)|$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{\infty} |\nu(E_j \cap A_k)|$$

$$\geq \sum_{j=1}^{n} \left| \sum_{k=1}^{\infty} \nu(E_j \cap A_k) \right|$$

$$\sum_{k=1}^{\infty} \nu(E_j \cap A_k) = \nu\left(E_j \cap \left(\bigcup_{k=1}^{\infty} A_k\right)\right)$$

$$= \nu(E_j)$$

($\because \nu$ is a complex measure)

$$\Rightarrow \sum_{k=1}^{\infty} |\nu|(A_k) \geq \sum_{j=1}^{n} |\nu(E_j)|$$

taking sup

$$\Rightarrow \sum_{k=1}^{\infty} |\nu|(A_k) \geq \sup\left\{ \sum_{j=1}^{n} |\nu(E_j)| \;\middle|\; \{E_i\} \subseteq \left(\bigcup_{k=1}^{\infty} A_k\right) \right\}$$

$$|\nu|\left(\bigcup_{i=1}^{\infty} A_i\right) \quad (\because (1))$$

$$\Rightarrow \sum_{k=1}^{\infty} |\nu|(A_k) \geq |\nu|\left(\bigcup_{k=1}^{\infty} A_k\right) \quad \text{——②}$$

from ①, ② $\sum_{k=1}^{\infty} |\nu|(A_k) = |\nu|\left(\bigcup_{k=1}^{\infty} A_k\right)$

2. $\forall E \in \mathcal{A}$ $|\nu|(E) < \infty$

we know that $\nu : \mathcal{A} \longrightarrow \mathbb{C}$

i.e $|\nu(E)| \in [0, \infty)$ $\forall E \in \mathcal{A}$ ——①

if $\exists E \in \mathcal{A}$ s.t $|\nu|(E) = \infty$ ——②

then let $E = E_0$

i.e $|\nu|(E_0) = \infty$ and $|\nu(E_0)| > 0$

then as $|\nu|(E_0) = \sup\{ |\nu(A)| + |\nu(B)| \mid A \cup B = E_0, A \cap B = \phi \}$

$\exists A \subseteq E_0$ s.t

$|\nu(A)| \geq 1 + |\nu(E_0)|$ ($\because |\nu|(E_0) = \infty$)

then $|\nu(E_0 \setminus A)| = |\nu(E_0) - \nu(A)|$

$\geq |\nu(A)| - |\nu(E_0)|$

≥ 1

$\Rightarrow |\nu|(A) + |\nu|(E_0 \setminus A) = |\nu|(E_0) = \infty$

if $|\nu|(A) = \infty$
then let $E_1 = A$

if $|\nu|(E_n \setminus A) = \infty$
then let $E_1 = E_0 \setminus A$

then $E_1 \subseteq E_0$ is trivial and

$|\nu|(E_1) = \infty$

$|\nu(E_1)| \geq 1$

similarly $E_2 \supseteq E_3 \supseteq \cdots$ construction s.t

$|\nu(A)| \geq n+1 |\nu(E_n)|$

i.e $|\nu|(E_{n+1}) = \infty$

$|\nu(E_{n+1})| \geq n+1$

now, $\left|\nu\left(\bigcap_{n=0}^{\infty} E_n\right)\right| = \lim_{n\to\infty} |\nu(E_n)| \geq \lim_{n\to\infty} n$

$\Rightarrow \left|\nu\left(\bigcap_{n=0}^{\infty} E_n\right)\right| = \infty$

but this contradicts ①

so, ② is false, i.e

$\forall E \in \mathcal{A}, |\nu|(E) < \infty$

4. $(\Omega, \mathcal{A})$ $M(\Omega)$ = all complex measures on $(\Omega, \mathcal{A})$

$\|\nu\| := |\nu|(\Omega) \quad \forall \nu \in M(\Omega)$

$\|\cdot\|$ denotes a norm on $M(\Omega)$ and $(M(\Omega), d)$ is complete metric space

d is metric induced by $\|\cdot\|$

$d(\mu, \nu) = \|\mu - \nu\| \quad \forall \mu, \nu \in M(\Omega)$

now $\|\nu\| \geq 0$ is trivial, also if $\|\nu\| = 0$ then

$|\nu|(\Omega) = 0$

$\Rightarrow \forall E \in \mathcal{A}\ |\nu(E)| \leq |\nu|(E) \leq |\nu|(\Omega) = 0$

$\Rightarrow |\nu(E)| = 0$

$\Rightarrow \nu(E) = 0$

i.e $\nu \equiv 0$ ← Partition of Ω

also $\|c\nu\| = |c\nu|(\Omega) = \sup\{\Sigma |c\nu(E_j)|\}$

$$= |c| \sup \left\{ \sum |\nu(E_j)| \right\}$$

$$= |c| \cdot \|\nu\|$$

for any $\{E_j\}$ partition $\sum |(\mu+\nu)(E_j)| \leq \sum |\nu(E_j)| + \sum |\mu(E_j)|$

$$\leq \|\mu\| + \|\nu\|$$

by taking sup

$$\Rightarrow \|\mu+\nu\| \leq \|\mu\| + \|\nu\|$$

$\Rightarrow \|\cdot\|$ is a norm on $M(\Omega)$

now, let $\{\nu_n\}_{n=1}^{\infty}$ be a cauchy sequence in $M(\Omega)$, then for any set $E \in \mathcal{A}$ $|\nu_n(E) - \nu_m(E)| \leq |\nu_n - \nu_m|(E) \leq \|\nu_n - \nu_m\|$

$\Rightarrow \{\nu_n(E)\}_{n=1}^{\infty}$ is cauchy in $\mathbb{C}$

$\Rightarrow \{\nu_n(E)\}_{n=1}^{\infty}$ is convergent in $\mathbb{C}$ ($\because \mathbb{C}$ is complete)

now let $\nu(E) = \lim_{n\to\infty} \nu_n(E)$

now for $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t $\forall\, n, m \geq N$

$$\|\nu_n - \nu_m\| < \varepsilon \quad (\because \{\nu_n\}_{n=1}^{\infty} \text{ is cauchy})$$

let $E \in \mathcal{A}$ and $\{E_1, E_2, \ldots E_k\}$ be a finite partition of E

for $n, m \geq N$

$$\sum_{j=1}^{k} |\nu_n(E_j) - \nu_m(E_j)| \leq \|\nu_n - \nu_m\| < \varepsilon$$

$$\Rightarrow \lim_{m\to\infty} \sum_{j=1}^{k} |\nu_n(E_j) - \nu_m(E_j)| < \varepsilon$$

($\because$ finite sum, so limit inside)

$$\Rightarrow \sum_{j=1}^{k} |\nu_n(E_j) - \nu(E_j)| < \varepsilon$$

$$\Rightarrow \sup\left(\sum_{j=1}^{k} |\nu_n(E_j) - \nu(E_j)| \right) < \varepsilon$$

$$\Rightarrow |\nu_n - \nu|(E) \leq \varepsilon \quad \forall\, n \geq N \quad (\because (1))$$

$$\Rightarrow |\nu_n - \nu|(\Omega) \leq \varepsilon \quad (\because E = \Omega)$$

then $\varepsilon \to 0$

$$\Rightarrow |\nu_n - \nu|(\Omega) \to 0$$

$$\Rightarrow \|\nu_n - \nu\| \longrightarrow 0 \text{ as } n \to \infty$$

now as $\|\nu_n - \nu\| < \varepsilon$ for $n \geq N$

$$\Rightarrow \|\nu\| < \infty \Rightarrow \nu \in M(\Omega) \quad (\because M(\Omega) \text{ is a vector space})$$

$\Rightarrow \nu_n \to \nu \in M(\Omega)$

$\Rightarrow M(\Omega)$ is complete

Tutorial-12 :

2. μ is a finite measure on $\mathcal{H}$

$\varphi : L^1(\Omega) \to \mathbb{C}$ bounded linear functional

$\exists\, g \in L^\infty(\Omega)$ s.t

$$\varphi(f) = \int_\Omega f g \, d\mu \quad \forall\, f \in L^1(\Omega)$$

as μ is finite $\Rightarrow$ μ is σ-finite

φ is a bounded linear functional

then let

$\nu(E) = \varphi(\chi_E)$ for $E \in \mathcal{H}$ as $\chi_E \in L^1(\Omega)$

$\nu(\phi) = \varphi(\chi_\phi) = \varphi(0) = 0$ ($\because \varphi : L^1(\Omega) \to \mathbb{C}$ is bounded linear functional)

$\Rightarrow \nu(\phi) = 0$ —— ①

now, let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ be disjoint then

$$\sum_{i=1}^{n} \chi_{E_i} \to \chi_{\bigcup_{i=1}^{\infty} E_i} \text{ in } L^1(\Omega) \text{ as } n \to \infty$$

($\because$ dom in conv

so $\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu(E_i)$ — ②

from ①, ② ν is a complex measure

and if $\mu(E) = 0$ i.e $\int_\Omega |\chi_E|^1 d\mu = |\mu(E)| = 0$

$\Rightarrow \chi_E = 0$ in $L^1(\Omega)$

then $\nu(E) = \varphi(0) =$

$\Rightarrow \nu \ll \mu$

then by Radon-Nikodym theorm, $\exists\, g \in L^1(\Omega)$ s.t

$$\nu(E) = \int_E g \, d\mu \quad \forall\, E \in \mathcal{H}$$

$$\nu(E) = \int_\Omega \chi_E g \, d\mu$$

let $f = \sum_{i=1}^{n} a_i \chi_{E_i}$ then $\sum_{i=1}^{n} c_i \nu(E_i) = \int_\Omega c_i \chi_E g \, d\mu$

$\Rightarrow \varphi(f) = \int_\Omega f g \, d\mu$

$g \in L^1(\Omega)$ and f is a basic fution

now, every bounded measurable function can be approximated using basic functions $\exists g \in L^1(\Omega)$ s.t

$$\forall f \in L^\infty(\Omega) \quad \varphi(f) = \int_\Omega f g \, d\mu$$

and as $L^1(\Omega)$ is dense in $L^\infty(\Omega)$ as

for any $f \in L^1(\Omega)$ let $f_n(x) = \begin{cases} f(x) & ; |f(x)| \leq n \\ n \frac{f(x)}{|f(x)|} & ; |f(x)| > n \end{cases}$

then $f_n \in L^\infty(\Omega) \ \forall \ n \in \mathbb{N}$

and $\|f_n - f\|_1 = \int_\Omega |f_n(\omega) - f(\omega)| \, d\mu$

$$= \int_{\{x \,|\, |f(x)| > n\}} |f_n(\omega) - f(\omega)| \, d\mu$$

$$= \int_{\{x \,|\, |f(x)| > n\}} \left| n \frac{f(x)}{|f(x)|} - f(x) \right| d\mu$$

$$\leq \int_{\{x \,|\, |f(x)| > n\}} \left(n - |f(x)| \right) d\mu$$

$$\leq \int_{\{x \,|\, |f(x)| > n\}} |f(x)| \, d\mu$$

as $n \to \infty \quad \int_{\{x \,|\, |f(x)| > n\}} |f(x)| \, d\mu \to 0$

$\Rightarrow \|f_n - f\|_1 \to 0$ as $n \to \infty$

so, $\{f_n\}_{n=1}^\infty$ s.t $\|f_n - f\|_1 \to 0$

$\Rightarrow$ every $L^1(\Omega)$ function can be approximated by $\{f_n\}_{n=1}^\infty \subseteq L^\infty(\Omega)$

$$\Rightarrow \varphi(f) = \int_\Omega f g \, d\mu \quad f \in L^1(\Omega)$$

now, $\|\varphi(f)\| \leq \|\varphi\| \|f\|_1$

$\forall E \in \mathcal{F}$

$$|\nu(E)| = |\varphi(\chi_E)| \leq \|\varphi\| \|\chi_E\|_1 = \|\varphi\| \, \mu(E)$$

$$\Rightarrow \left|\int_E g\,d\mu\right| \leq \|g\|\,\mu(E)$$

$$\Rightarrow \left|\int_\Omega g\,d\mu\right| \leq \|g\|\,\mu(\Omega) = M$$

So, $\exists M \in \mathbb{R}_{\geq 0}$ s.t $\left|\int_\Omega g\,d\mu\right| \leq M \Rightarrow |g(x)| \leq M$ a.e

$$\Rightarrow \|g\|_\infty \leq M$$

$$\Rightarrow g \in L^\infty(\Omega)$$

So, $\exists g \in L^\infty(\Omega)$ s.t $\varphi(f) = \int_\Omega fg\,d\mu \quad \forall f \in L^1(\Omega)$

1. $\{f_n\}_{n=1}^\infty \quad f_i : \Omega \to \mathbb{R} \quad \int_\Omega f_i\,d\lambda < \infty \ \forall i \in \mathbb{N}$

s.t $f_n \to 0$ $n \to \infty$ uniformly on every compact set

but $\int_{\mathbb{R}} f_n\,d\lambda = 1 \ \forall n \in \mathbb{N}$

$$f_n(x) = \begin{cases} \frac{1}{n} ; & 0 \leq x \leq n \\ 0 ; & \text{otherise} \end{cases}$$

$\int_{\mathbb{R}} f_n\,d\lambda = \frac{1}{n}.n = 1$ and seq conv uniformly to 0

as $\sup_{x \in K} |f_n(x)| \leq \frac{1}{n} \to 0$

2. $\{f_n\}_{n=1}^\infty \quad f_i : (0,1) \to \mathbb{R}$ and $\int_{(0,1)} f_i\,d\lambda < \infty \quad \forall i \in \mathbb{N}$

$f_n(x) \to 0$ as $n \to \infty \quad \forall x \in (0,1)$

$\int_{(0,1)} f_n\,d\lambda \to \infty$ as $n \to \infty$

let $f_n(x) = \begin{cases} n^2 ; & 0 < x < 1/n \\ 0 ; & 1/n \leq x < 1 \end{cases}$

then trivial to see that $\forall x \in (0,1) \ \exists N \in \mathbb{N}$ s.t

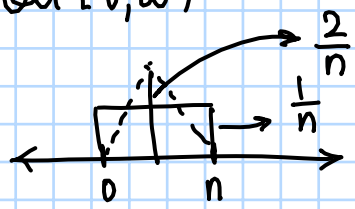
$$\frac{1}{N} \leq x$$

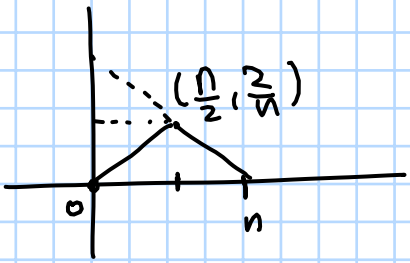
$$\Rightarrow f_n(x) = 0 \quad \forall n \geq N$$

$$\Rightarrow f_n(x) \to 0 \text{ as } n \to \infty$$

and $\int_{(0,1)} f_n(x)\,dx = \int_0^{1/n} n^2\,dx = n \to \infty$ as $n \to \infty$

3. $\{f_n\}_{n=1}^{\infty}$ sequence of real valued cont. Lebesgue integrable functions on $[0,\infty)$





let $f_n(x) = \begin{cases} \frac{4}{n^2}x ; & x \in [0, \frac{n}{2}] \\ \frac{4}{n} - \frac{4}{n^2}x ; & x \in (\frac{n}{2}, n] \\ 0 & ; \text{otherwise} \end{cases}$

then trivial to see $f_n(x) \to 0$ uniformly

as $\sup_{x \in [0,\infty)} |f_n(x)| \leq \frac{2}{n} \to 0$ as $n \to \infty$

also $\forall n \in \mathbb{N}$, f_n cont & Lebesgue integrable

but $\int_{[0,\infty)} |f_n - f| \, d\lambda = \int_0^{\infty} f_n(x) dx = 1$

$\Rightarrow \lim_{n\to\infty} \int_{[0,\infty)} |f_n - f| \, d\lambda = 1$

4. $f : (0,1) \longrightarrow \mathbb{R}$ is Lebesgue integrable

$$\lim_{n\to\infty} \int_0^1 n \ln\left(1 + \frac{|f(x)|^2}{n^2}\right) dx$$

let $f_n(x) = n \ln\left(1 + \frac{|f(x)|^2}{n^2}\right)$

$$f_{n+1}(x) = (n+1) \ln\left(1 + \frac{|f(x)|^2}{(n+1)^2}\right)$$

let $a_i = \left(1 + \frac{a}{n}\right)$ $\quad i = 1, 2 \ldots n \quad a_{n+1} = 1$

then $(a_1 a_2 \ldots a_{n+1})^{\frac{1}{n+1}} \leq \frac{(n)(a_1) + 1(a_{n+1})}{n+1}$

$$\Rightarrow \left(1 + \frac{a}{n}\right)^{\frac{n}{n+1}} \leq \frac{n + a + 1}{n+1}$$

$$\Rightarrow \left(1 + \frac{a}{n}\right)^n \leq \left(1 + \frac{a}{n+1}\right)^{n+1}$$

so, $f_{n+1}(x) \geq f_n(x) \quad \forall x \in (0,1)$

also, for $|f(x)| \neq \infty$ (a.e on $(0,1)$ as integrable)

$$\lim_{n\to\infty}\left(1+\frac{|f(x)|^2}{n^2}\right)^n = e^{\lim_{n\to\infty}\frac{|f(x)|^2}{n^2}\cdot n}$$

$$= e^0 = 1$$

$$\Rightarrow \lim_{n\to\infty} \text{ or } \left(1+\frac{|f(x)|^2}{n^2}\right)^n = 0 \quad \text{a.e}$$

by MCT $\lim_{n\to\infty}\int_{(0,1)} f_n d\lambda = \int_{(0,1)} f\, d\lambda = \int_{(0,1)} 0.d\lambda = 0$ (↓ as a.e)

6. $\left(1-\frac{x}{n}\right)^n e^{x/3}\,\chi_{(0,n)}$

for $x \in (n, n+1)$ $\quad 0 < \left(1-\frac{x}{n+1}\right)^{n+1} e^{x/3}\,\chi_{(n,n+1)}$

$\Rightarrow f_n(x) \leq f_{n+1}(x) \quad \forall x \in (n,\infty)$

for $x \in (0,n)$ $\quad f_n(x) = \left(1-\frac{x}{n}\right)^n e^{x/3}$

$$f_{n+1}(x) = \left(1-\frac{x}{n+1}\right)^{n+1} e^{x/3}$$

then as $1-\frac{x}{n} > 0,\ 1-\frac{x}{n+1} > 0$

$$\Rightarrow \left(1-\frac{x}{n}\right)^n < \left(1-\frac{x}{n+1}\right)^{n+1}$$

so, $\forall x \in (0,\infty)\ f_n(x) \leq f_{n+1}(x)$

and $\lim_{n\to\infty} f_n(x) = e^{\lim_{n\to\infty} -x/n \cdot n}\, e^{x/3}$

$$= e^{-x} e^{x/3}$$

$$= e^{-2x/3}$$

$$\Rightarrow \lim_{n\to\infty}\int_0^\infty f_n(x)dx = \int_0^\infty e^{-2x/3}\,dx$$

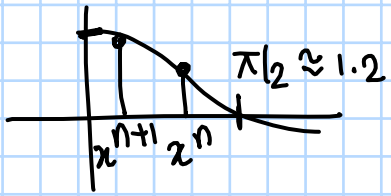
$$= \left|\frac{e^{-2x/3}}{-2/3}\right|_0^\infty$$

$$= 0 + \frac{e^0}{+2/3} = \frac{3}{2}$$

7. $\lim_{n\to\infty} \int_{[0,1]} \cos(x^n)\, d\lambda(x)$

$f_n(x) = \cos(x^n) \quad x \in (0,1)$

then $x^{n+1} < x^n$



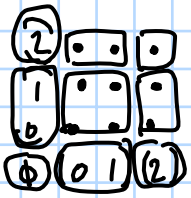
$\cos(x^{n+1}) > \cos(x^n)$

$\Rightarrow f_n(x) \leq f_{n+1}(x) \; \forall x \in [0,1]$

also $\lim_{n\to\infty} f_n(x) = \begin{cases} 1; & x \in [0,1) \\ \cos(1); & x = 1 \end{cases} = f(x)$

$\Rightarrow \lim_{n\to\infty} \int_{[0,1]} f_n(x)\,dx = \int_0^1 f(x)\,dx = \int_0^1 1\,dx = 1$

1. complete measure spaces



$\mu_1 = \mu_2$

$\mathcal{A}_1 = \mathcal{A}_2 = \{\phi, \{0,1,2\}, \{0,1\}, \{2\}\}$

$\mu_1(\phi) = 0$

$\mu_1(\{2\}) = 0$

$\mu_2(\{0,1\}) = 1$

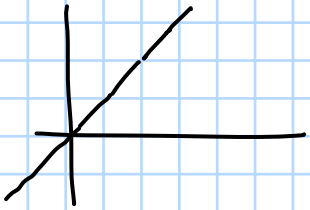
now, $\{2\} \times \{0,1\} \in \mathcal{A}_1 \otimes \mathcal{A}_2$

$\mu(\{2\} \times \{0,1\}) = 0$

but $\{2\} \times \{0\} \notin \mathcal{A}_1 \otimes \mathcal{A}_2$

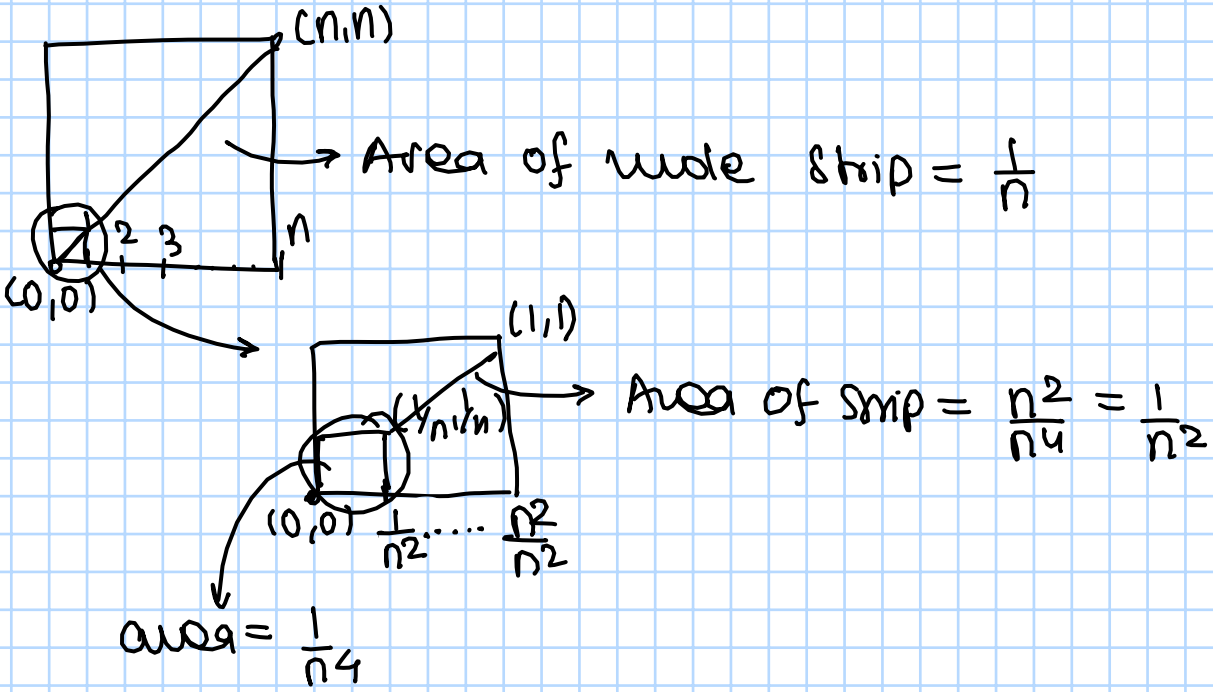
2. $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \lambda_2)$

$D = \{(x,y) \in \mathbb{R}^2 \mid x = y\} \in \mathcal{B}(\mathbb{R}^2)$



$\mathcal{B}(\mathbb{R}^2) = \mathcal{P}(A)$ ← set of all open sets of $\mathbb{R}^2$

also we already know that a closed square $\in \mathcal{B}(\mathbb{R}^2)$



tell as $n \to \infty$, area $\to 0$ and so

$\cap A_n \to 0 \Rightarrow D \in \mathcal{B}(\mathbb{R}^2)$

↳ this special strip

and $\lambda(D) = 0$

3. $\{x_{m,n} \in \mathbb{R} \mid m,n \in \mathbb{N},\ x_{m,n} \geq 0\}$

let $A = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{m,n}$

$B = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} x_{m,n}$

for any M, N: $\sum_{m=1}^{M} \sum_{n=1}^{N} x_{m,n} = \sum_{n=1}^{N} \sum_{m=1}^{M} x_{m,n} \leq B$

$\Rightarrow A \leq B$

similarly $B \leq A$

4. $g: [0,1] \to \mathbb{R}$

$f: [0,1] \times [0,1] \to \mathbb{R}$ $\quad f(x,y) = 2g(x) - 3g(y)$ is integrable over $[0,1] \times [0,1]$

f integrable

$\Rightarrow |f|$ integrable

$\Rightarrow -f$ integrable

so, $\int_{[0,1]\times[0,1]} -f \, d\lambda < \infty$

$$\Rightarrow \int_0^1 \left(\int_0^1 3g(y) - 2g(x)\, dx \right) dy < \infty \quad (\because \text{fubini's})$$

$$\Rightarrow \int_0^1 3g(y)\, dy - \int_0^1 2g(y)\, dy < \infty$$

$$\Rightarrow \int_0^1 g(y)\, dy < \infty$$

$\Rightarrow g$ is integrable